

ON THE SEPARATION OF PREFERENCES AMONG  
MARKED POINT PROCESS WAGER ALTERNATIVES

A Dissertation

by

JEE HYUK PARK

Submitted to the Office of Graduate Studies of  
Texas A&M University  
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2008

Major Subject: Industrial Engineering

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## ABSTRACT

On the Separation of Preferences among Marked

Point Process Wager Alternatives. (May 2008)

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Chair of Advisory Committee: Dr. Martin A. Wortman

A wager is a one time bet, staking money on one among a collection of alternatives having uncertain reward. Wagers represent a common class of engineering decision, where “bets” are placed on the design, deployment, and/or operation of technology. Often such wagers are characterized by alternatives having value that evolves according to some future cash flow. Here, the values of specific alternatives are derived from a cash flow modeled as a stochastic marked point process. A principal difficulty with these engineering wagers is that the probability laws governing the dynamics of random cash flow typically are not (completely) available; hence, separating the gambler’s preference among wager alternatives is quite difficult.

In this dissertation, we investigate a computational approach for separating preferences among alternatives of a wager where the alternatives have values that evolve according to a marked point processes. We are particularly concerned with separating a gambler’s preferences when the probability laws on the available alternatives are not completely specified.

To Jee Hye, Jared and Joel

## TABLE OF CONTENTS

CHAPTER		Page
I	INTRODUCTION . . . . .	1
	A. Research Objectives . . . . .	2
	B. Approach . . . . .	4
	C. Dissertation Organization . . . . .	4
II	LITERATURE REVIEW . . . . .	6
	A. Wagers in Engineering . . . . .	6
III	PREFERENCE SEPARATION AMONG SPECIAL WAGER ALTERNATIVES . . . . .	8
	A. Marked Point Process Wager Alternatives . . . . .	9
	B. Expected Utility Theorem . . . . .	11
	C. Expected Utilities of Special Wager Alternatives . . . . .	13
	1. Sequence of i.i.d. random variable rewards . . . . .	15
	2. Compound renewal process rewards . . . . .	16
	3. Irreducible positive recurrent Markov chain rewards . . . . .	19
	4. Zero mean martingale rewards . . . . .	23
	5. Stationary ergodic process rewards . . . . .	27
	D. Preference Ordering of Special Wager Alternatives . . . . .	29
IV	PARTITIONING WAGER ALTERNATIVES UNDER NON- UNIQUE RISK CHARACTERIZATION . . . . .	32
	A. Mathematical Formulation . . . . .	33
	B. Construction of Feasible Risk Sets . . . . .	36
	1. Bounded marginal distribution . . . . .	37
	2. Bounded conditional distribution . . . . .	38
	3. Ordinal distributions . . . . .	38
	4. Independence . . . . .	39
	C. Preference Ordering among Wager Alternatives under Non-unique Risk Characterization . . . . .	41
V	NUMERICAL APPROACH FOR SEPARATION OF PREF- ERENCES AMONG WAGER ALTERNATIVES . . . . .	46

CHAPTER	Page
A. Construction of Mathematical Programming Problems . .	48
B. A Solution Method for Optimization Problems in Wager .	52
1. An algorithm for feasible solution exploration . . . .	53
2. An algorithm for objective function evaluation . . . .	56
C. Identification of Expected Utility Bounds . . . . .	58
D. Wagering with MCMC Optimization Algorithm . . . . .	60
E. An Example: Separation of Wager Alternatives Using MCMC Optimization . . . . .	61
VI CONCLUSION . . . . .	66
A. Future Research . . . . .	68
1. Sensitivity analysis . . . . .	68
2. On stationary marked point process rewards . . . . .	69
B. Further Areas for Application . . . . .	70
REFERENCES . . . . .	72
APPENDIX A . . . . .	78
APPENDIX B . . . . .	81
APPENDIX C . . . . .	84
APPENDIX D . . . . .	86
APPENDIX E . . . . .	89
APPENDIX F . . . . .	92
APPENDIX G . . . . .	94
VITA . . . . .	100

## LIST OF FIGURES

FIGURE		Page
1	A realization of rewards by selecting a wager alternative. . . . .	9
2	Sample paths from a Markov chain having a $N(0.2, 0.15^2)$ invariant distribution. . . . .	23
3	Boundaries a, b, c and d for interval $I_\alpha = [l_\alpha, u_\alpha]$ such that $l_\alpha \in [a, b]$ and $u_\alpha \in [c, d]$ . . . . .	60
4	Distributions and the corresponding boundaries on the expected utility for wager alternative $\alpha$ . . . . .	64
5	Marginal distributions and the corresponding boundaries on the expected utility for wager alternative $\beta$ . . . . .	65
6	Cumulative distributions of the solutions for alternative $\alpha$ . . . . .	96
7	Cumulative distributions of the solutions for alternative $\beta$ . . . . .	99

## CHAPTER I

### INTRODUCTION

Design, deployment and operation of technology in engineered systems all require choosing from among alternatives. The value of any alternative typically evolves over time as an uncertain cash flow. The probability law on a cash flow process determines the net present value of the alternative; often these probability laws are not completely specified. The analysis of the alternatives leading to a weak preference ordering is a very difficult task. From the decision maker's perspective, choosing from among alternatives that have a random cash flow is a wager. The terminology "wager" is used to describe the activity of betting on an alternative having uncertain outcome. This perspective is held in making decisions in engineered systems. However, the term is sometimes used synonymously with *game* [1]. To avoid confusion between a wager and a game, we define a "wager" as a one-time bet on an alternative having an outcome that cannot be predicted with certainty, while a "game" describes a sequence of betting decisions involving multiple players where a known probability governs all outcomes (e.g. poker or casino games). A wager involves a one opportunity; generally the potential loss can approach the gambler's total wealth (called *a high-stakes wager*). High-stakes wagering clearly requires considerable care. We define a wager on engineered systems to be an *engineering wager*.

Decision and game theories that characterize uncertainty with probability on events have been well developed. Conventional decision problems with uncertainty use a specified probability measure on the  $\sigma$ -algebra generated by a set of possible outcomes for each alternative. Formulating probability laws to capture the uncertainty

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The journal model is *IEEE Transactions on Automatic Control*.



on outcomes and constructing a set of probability measures subject to probability laws are difficult. In our research, we explore certain marked point processes suitable to capture dynamics of alternative's random cash flow. Marked point process representations are useful because the values of wager alternative are often acquired over time. The value of an alternative is then a function of the marked point process dynamics. From this standpoint, the engineering wager often can be thought of as a wager having a marked point process representation for each decision alternative. We call such decision scenarios *marked point process wager*. Since the probability law on a wager alternative is its *risk*, selecting the best wager alternative under conditions of uncertainty is synonymous with identifying the alternative having the most preferred risk.

In this research, we first explore the risk associated with a marked point process wager alternative. Then wager alternatives can be compared by preferences for the risks using the expected utility theorem. We present principles and computational frameworks for separating marked point process risks based on preference and for finding the most preferred alternatives. The framework includes optimization formulations and the solution algorithm for identifying the expected utilities of the alternatives and making them according to preferred risk.

#### A. Research Objectives

We focus our research on marked point process wagers, that is, one-time decisions that have marked point process models for rewards associated with alternatives. The objectives of this research are to develop a risk assessment procedures for marked point process wager alternatives, to construct principles which support ordering preferences

for such alternatives, and to explore a computational method to support preference ordering. The computational framework must enable the identification of the most favorable risk among the wager alternatives. The research examines both general marked point process models and special stochastic process models for the values of wager alternatives.

For special marked point process reward streams, we consider the sequence of *i.i.d.* random variables in discrete time, compound renewal processes, irreducible positive recurrent Markov Chains, zero-mean martingales, and zero-mean square-integrable stationary ergodic processes. For general process rewards alternatives, where there is limited information on their rewards, we develop the variational optimization formulation to support optimal wagering.

In constructing preference separation principles and computational frameworks among marked point process wager alternatives, we address the following objectives:

- Represent cash flow as marked point process wager alternatives so as to compare them.
- Relaxation the requirement of a complete weak ordering among wager alternatives by preference.
- Separate the alternatives on the basis of risk preference.
- Exploit characteristics of special stochastic processes to find corresponding risks when the wager presents cash flows of the special forms.

## B. Approach

In general risk theory, the risk of an alternative is the distribution function of the net present value (NPV) of that alternative. The expected utility theorem provides the foundation on which the number of alternatives can be ordered by preference. We accept this normative theorem throughout this dissertation. For general situation when information on cash flow is not a priori available, we can capture the risk of such marked point process wager alternative within a family of distribution functions the NPV associated with the process can have.

We construct a computational framework obtaining expected utilities of the risk encumbered alternatives and develop principles that support the preference ordering among respective alternatives. The framework and principles together show that it is often possible to find the most preferred alternative even a unique probability law on cash flows is not available. The framework appears as *a constrained calculus of variation* formulation. We suggest a solution method using Monte Carlo integration that is applicable for all such programs. To engage the sets of expected utilities, interval graph representation is employed for developing the principles. For each type of marked point process wager alternative, an example is illustrated to show how the risk of the alternative can be obtained and how its value is calculated. The last example, in particular, shows the application of principles for the separation of preferences among wager alternatives.

## C. Dissertation Organization

The remainder of this dissertation is organized into five chapters. Chapter II offers a review of the literature that discusses engineering wagers. Chapter III provides

representation of the alternatives for wagers available for quantitative value analysis and also discusses the Expected Utility Theorem and its application to marked point process wager alternatives. In addition, special forms of marked point process wager alternatives, with corresponding statistical inferences, are considered in this chapter. Development of the computational framework for risk assessment of general stochastic process wager alternatives is presented in Chapter IV. This chapter also provides a useful lemma and a corollary for separation of the risk preferences. In Chapter V, an approximation method for the presented mathematical programming in the framework is presented with examples. Finally, we conclude this dissertation with a discussion of real world settings where our conclusions are applicable as well as some future topics for extending our research.

## CHAPTER II

### LITERATURE REVIEW

#### A. Wagers in Engineering

In engineering, a wager means *staking money on an irrevocable decision to select from among engineered system alternatives*. Since engineering design generally involves a number of elements about which only limited information is available, design selection is generally risk encumbered. Here, we focus on the literature related to probabilistic approaches for engineering design problems in which decisions are made based on risk preference.

Most credible research in this area adopts the perspective of Koopman [2, 3], Good [4], Ramsey [5], and Savage [6] on probability. Formulating probability laws to capture the uncertainty associated with choice under risk is difficult. Using the concept of *utility*, first introduced by Bernoulli [7], Von Neumann and Morgenstern [8] published a work that formed the foundation of game theory and, in the process, exposed a formulation that enables us to order risks linearly by preference. They proved that *expected utility* exists under a very reasonable set of conditions (or *axioms*). Further, expected utility is the only measure that is consistent with the decision maker's preferences. Although axioms underlying their theory and their extensions have been studied intensively [9, 10, 11, 12], the basic expected utility paradigm has remained unchanged. Weber et al. have described recent developments in game theory in their survey paper [13].

Fishburn [14] explored the use of incomplete information on probability when comparing alternative strategies in a typical formulation of expected utility theory. He categorized incomplete information on probability into four classes: no informa-

tion; ordering of probabilities; sets of inequalities; and bounded probabilities. These ‘*imprecise*’ measures of probability are used to determine an ordering or partial ordering of the expected utilities of alternative strategies. The criterion for selection of a strategy is the maximization of the expected utility. Further research utilizes these types of probability measures [15, 16].

Weisman and Holzman [17] suggested an unconstrained nonlinear programming technique for solving engineering design problems. They considered two types of distribution functions, normal and non-normal, for obtaining expected utility and utility functions derived from relations between enterprise costs and revenues. In the normal case, the optimal solution containing the best design is found, while in the non-normal case, only an upper bound is suggested. Recent research has been focused more on the application of such methods to practical problems in many areas [18, 19].

Several limitations uncovered by previous research have not yet been adequately resolved. These limitations include the following:

- The random rewards provided by a design have rarely been considered. Even in the random case, the discrete distribution functions are generally assumed to be known, and the general distribution function that captures the dynamics of possible random rewards has not been derived. With unknown distributions, no proper approach has been developed for a choice of alternatives. The marked point process rewards representation basically accounts for the random rewards of wager alternatives.
- No study has been conducted to develop an analytical framework for selecting the most favorable design when information about the rewards for the wager alternatives is restrictive.

## CHAPTER III

### PREFERENCE SEPARATION AMONG SPECIAL WAGER ALTERNATIVES

In this chapter, we consider wagers whose alternative rewards are well-known point processes. Because the risk is represented as the distribution function on the NPV of the rewards, it is possible to compare wager alternatives based on preferences for respective risks if we can obtain such distribution functions. In the situation that there are only a small number of rewards for a wager alternative, it is easy to obtain the risk. In other cases, the acquisition of the risk for the marked point process wager alternatives is generally difficult. We investigate some of the marked point process rewards which hold a property that makes risk simpler to formulate numerically. This characteristic might provide a computational advantage for obtaining risk. We then find the expected utilities for such marked point process wager alternatives in order to select the most favorable one among them.

In section A, the risk representation associated with a marked point process wager alternative is presented. This section introduces a formal definition of the risk of a wager alternative. Since the Expected Utility Theorem (EUT) provides a mapping from the risk of an alternative to a real value, called *the expected utility*, this theorem is essential for ordering alternatives according to preferences. In section B, we describe the EUT and explain its supporting axioms as applied to a wagering problem. The general formulation of the expected utilities for marked point processes wager alternatives is also introduced. Section C illustrates wager alternatives whose rewards form special marked point processes. The characteristics of such processes that are necessary for obtaining risks are presented in detail. Even with these risks known, it still might be hard to analytically calculate the expected utilities. For each type of rewards, a mathematical formulation and an estimator for the expected utility are

developed, together with a set of examples. The last section accounts for the preference ordering of wager alternatives with certain special marked point process rewards.

#### A. Marked Point Process Wager Alternatives

The rewards associated with certain wager alternatives are accumulated over time. Hence, the reward process can be considered as a cash flow with revenues and costs. The costs include all of the expenses to be paid when deciding upon a wager alternative (e.g., raw material purchase costs, production costs, inventory costs, and expenditures for maintenance). The costs are represented as negative revenues during some time period  $t$ ; the revenues, i.e., product sales, are represented as positive rewards during period  $t$ . Figure 1 shows an example of a cash flow realization. Upward arrows depict revenues, and downward arrows describe costs. The downward arrow at time 0 implies an initial cost (e.g., a setup, service, launching cost).

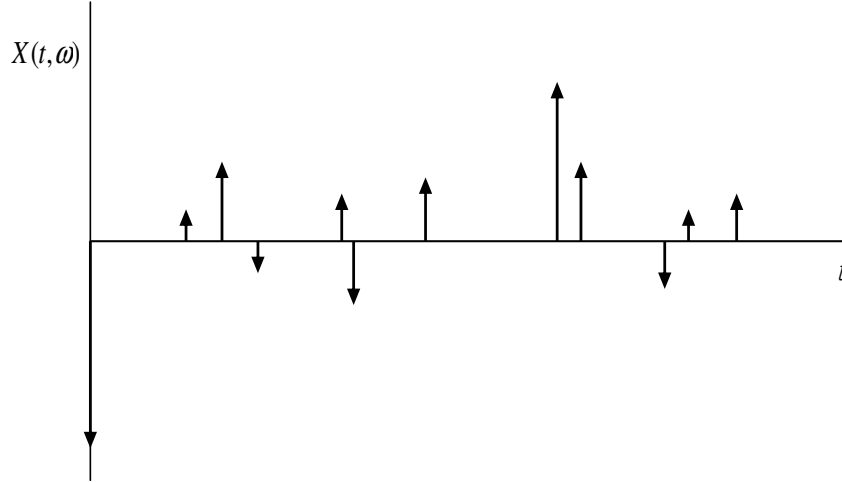


Fig. 1. A realization of rewards by selecting a wager alternative.

The rewards (cash flow) process associated with a given wager alternative can



rarely be predicted with certainty and, therefore, must be modeled as a stochastic process. Choosing the most preferred alternatives for a wager implies selecting the alternative cash flow with the most favorable risk. Since the *risk* of an alternative is defined by *the cumulative distribution function corresponding to the present value of the alternative*, the distribution function (or risk) of a wager alternative provides the means for comparing wager alternatives [20, 21]. In other words, the most preferred wager alternative can be selected by comparing the distribution functions of the Net Present Value (NPV) for the marked point process rewards of the alternatives.

For a fixed alternative  $\alpha \in \mathcal{T}$ , where  $\mathcal{T}$  is the index set for alternatives of a given wager with all random elements defined on the probability space  $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ , let  $\{(X_n^\alpha, T_n^\alpha) : n \in \mathbb{Z}_+\}$  be a sequence of rewards by representing  $(X_k^\alpha, T_k^\alpha)$ ,  $k = 1, \dots, n$  as the amount and time of  $k$ th rewards for wager alternative  $\alpha$ . Let  $g$  be a discount function from  $\mathbb{R} \times \mathbb{R}_+$  to  $\mathbb{R}$ . The NPV, then, is the cumulative rewards discounted through the function  $g$ . Let  $Z_\alpha$  be the NPV of a given reward process, a measurable mapping from  $\mathbb{R} \times \mathbb{R}_+$  to  $\mathbb{R}$  defined by

$$Z_\alpha = \sum_{n=0}^{\infty} g(X_n^\alpha, T_n^\alpha).$$

The cumulative distribution function (or risk) corresponding to the NPV of a marked point process wager alternative  $\alpha$  is given by

$$F_\alpha(x) = P\{Z_\alpha \leq x\}.$$

The distribution function  $F_\alpha(z)$  characterizes the uncertainty of the value for alternative  $\alpha$ . Thus, it becomes the *risk* for given wager distribution  $\alpha$ .

## B. Expected Utility Theorem

A gambler's choice is *rational* if it leads him or her to the best outcome attainable in his or her situation while satisfying certain normative axioms. What is defined as *best* is derived from the order of preferences the gambler entertains regarding the outcomes that might be achieved. So, a wager in the rational sense implies the choice of the most favorable alternative with respect to reward.

Let  $\mathbf{X}$  be a support of the NPV of risk. And let  $D(\mathbf{X})$  be the collection of distribution functions on  $\mathbf{X}$ . Each wager alternative  $\alpha$  has a corresponding reward distribution function (or risk)  $F_\alpha \in D(\mathbf{X})$ . The rational gambler selects the alternative  $\alpha^*$  having the most favorable risk  $F_{\alpha^*} \in D(\mathbf{X})$ . In order to select the most favorable alternative  $\alpha^*$ , we appeal to the expected utility theorem. Von Neumann's Expected Utility Theorem (EUT) and its supporting axioms offer a foundation for building a measure by which wager alternatives can be compared in a common metric space [22]. For the purpose of this research, we accept the axioms underlying the EUT and the assumption that wagers are rational decisions.

### *Expected Utility Theorem (EUT)*

Let  $\succeq$  be a *binary relation* (or *preference relation*) on  $D(\mathbf{X})$ . There exists a continuous function  $u : \mathbf{X} \mapsto \mathbb{R}$  such that

$$F \mapsto \int_{\mathbf{X}} u(x) dF(x)$$

represents  $\succeq$  if, and only if, the *Weak Ordering*, *Continuity*, and *Independence* axioms are satisfied. Moreover, the function  $u$  is unique up to the positive affine transformations.

The EUT reveals the existence of a unique (up to the linear transformation) *utility function*  $u$  that results in a value or *expected utility* if, and only if, three

axioms are satisfied. In other words, the distribution function  $F$  is mapped into a value on the real line when we have a utility function; it derives from the following axioms:

The *Weak Ordering* axiom means that among the distribution functions of the NPV of rewards in  $D(\mathbf{X})$ , one is preferred over the others.<sup>1</sup> And the preference (or *risk preference*) is transitive for all distributions in  $D(\mathbf{X})$ . This implies that transitivity is a characteristic of rational behavior.

The *Continuity* axiom asserts that preference ordering holds for the limiting distributions. For example, let  $G_n$  be the sequence of distribution functions such that their limiting distribution is  $G$ . If  $H$  is preferred to all  $G_n$ , then  $H$  is preferred to  $G$ . If all  $G_n$  are preferred over  $H$ , then  $G$  is preferred over  $H$ . This asserts that inclusion of the limiting distribution function of the NPV of rewards in preference ordering is allowable as rational behavior.

The *Independence* axiom describes the invariance of preference ordering over probability mixtures with a third distribution function.<sup>2</sup> If a reward distribution  $F$  is preferred to a distribution  $G$ , then this preference is unchanged over mixtures of  $F$  and  $G$  with a third distribution  $H$ .

The three axioms together define *rational behavior*. Thus, by assuming that the gambler in the wager is rational, the gambler's preference on the risk is revealed when the EUT is applied.

Suppose that a marked point process  $\{(X_k^\alpha, T_k^\alpha) : k \leq n\}$  be the reward process for alternative  $\alpha$ . Let  $Z_\alpha$  be the NPV of a given reward process and  $F_\alpha$  be the risk

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<sup>1</sup> It means that the preference is both complete (i.e., for all  $F, G \in D(\mathbf{X})$ ,  $F \succeq G$  or  $G \succeq F$  holds.) and transitive (i.e., if  $F \succeq G$  and  $G \succeq H$  for any  $H \in D(\mathbf{X})$ ,  $F \succeq H$ ).

<sup>2</sup> For all  $F, G, H \in D(\mathbf{X})$  and all  $c \in [0, 1]$ ,  $F \succeq G$  implies  $cF + (1 - c)H \succeq cG + (1 - c)H$ .

associated with wager alternative  $\alpha$ . Then by the EUT, we have the expected utility associated with the alternative  $\alpha$  as follows,

$$E[u \circ Z_\alpha] = \int_{\mathbf{z}} u(z) dF_\alpha(z) \quad (3.1)$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}_+^n} (u \circ z)(\underline{y}) dG_\alpha(\underline{y}) \quad (3.2)$$

where  $G_\alpha(\underline{y})$ ,  $\underline{y} = (y_1, \dots, y_{2n})$ , is a joint distribution function  $G_\alpha(\underline{y}) = P\{X_1 \leq y_1, \dots, X_n \leq y_n, T_1 \leq y_{n+1}, \dots, T_n \leq y_{2n}\}$  for  $X_1, \dots, X_n, T_1, \dots, T_n$ . The formulation (3.2) is available because the risk  $F_\alpha$  has origins from the sequence of random variables  $(X_k^\alpha, T_k^\alpha)$  where  $X_k^\alpha$  implies the amount of the  $k^{th}$  reward while  $T_k^\alpha$  represents the time of the  $k^{th}$  reward. This shows that the formulation for the expected utility can be written in terms of a joint distribution function if the rewards of a wager alternative form a marked point process. The number of rewards in this formulation must not be finite. Because of the discount function, the affect of a reward far from the betting moment will be small enough to be ignored. The following section focuses on some special marked point process wager alternatives that have the properties necessary for obtaining expected utilities.

### C. Expected Utilities of Special Wager Alternatives

Generally, the risk of wager alternatives associated with marked point processes is difficult to obtain. However, certain marked point processes enjoy properties that make risk more accessible. The characteristics of such stochastic processes can offer certain computational advantages, which occasionally lead to simple computations of the expected utility. We consider some special marked point process wager alternatives whose rewards are expressed as follows:

- (a) a sequence of *i.i.d.* random variables in discrete time,
- (b) compound renewal process,
- (c) irreducible positive recurrent Markov Chain,
- (d) zero-mean martingale,
- (e) zero-mean, square-integrable stationary and ergodic process.

When a small number of rewards are disclosed within a specific time period, it is fairly easy to obtain the expected utilities through a calculation of the NPV and its distribution. In cases where a *large* number of rewards<sup>3</sup> exist, obtaining the risk and the expected utility often becomes a difficult task. Processes (c), (d), and (e) are examples of difficult scenarios. In these situations, the distribution of the NPV might be estimated by simulating its reward process. This leads us to consider variations of the *Central Limit Theorem (CLT)*, since 1) the discount function makes the magnitude of rewards diminish away from the moment of decision, and 2) the theorem includes a set of weak convergence results in probability theory for the large number of random variables. If the CLT can be applied for obtaining the risk for anticipated random rewards, we can then utilize the asymptotic normality of CLT. The existence of such a limiting distribution will be discussed in terms of the *Strong Law of Large Numbers (SLLN)* while investigating the mean and variance of a NPV normal random variable. The approximation methods for estimating mean and variance related to risk will be developed with examples. These methods are more computationally efficient than calculating the risks directly.

There are a variety of discount functions. The exponential form with parameter  $\delta$  has been widely used among these functions. We adopt this exponential discount function for investigating the NPV of rewards associated with the following special

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<sup>3</sup> A large enough number of rewards to allow the limit theorems, generally more than 20.

types of wager alternatives as long as the different type for each discount function is specified.

### 1. Sequence of i.i.d. random variable rewards

In this section, we consider discrete time *i.i.d.* random variables  $X_k$ ,  $k = 1, \dots, n$  for rewards from a wager alternative selection. The total rewards  $Z$  can then be computed by

$$Z = \sum_{k=1}^n e^{-\delta \cdot k} X_k$$

where  $\delta$  is a discount rate. Then, the expected utility associated with a risk  $F(z) = P\{Z \leq z\}$  is

$$\begin{aligned} E[u \circ Z] &= \int_{\mathbf{x}} u(z) dF(z) \\ &= \int_{\mathbb{R}^n} (u \circ z)(\underline{y}) dG(\underline{y}) \end{aligned}$$

where  $G(\underline{y})$  is a joint distribution function for  $X_1, \dots, X_n$  with

$$G(\underline{y}) = P\{X_1 \leq y_1, \dots, X_n \leq y_n\} = \prod_{i=1}^n P\{X_i \leq y_i\}$$

We can obtain the expected utility of the wager alternative that has the sequence of *i.i.d.* random rewards using either the distribution function of the net present value of total rewards or the joint distribution function of all random rewards.

### A wager alternative example with Gamma distributed rewards

Let  $X_k$ ,  $k = 1, \dots, 10$  be nonnegative *i.i.d.* random variables which represent the present value of rewards when alternative  $A$  is selected. Suppose that each reward  $X_k$  is Gamma distributed with a \$500,000 mean and a 0.5 scale parameter. Then,

the shape parameter  $\alpha_k = 1$  for all  $k$  because of the mean  $0.5 = \alpha_k \beta$ . Thus, NPV  $Z_A$  of these rewards has a Gamma distribution with parameters 10 and 0.5.

$$Z_A = \sum_{k=1}^{10} X_k \sim \text{Gamma}\left(\sum_{k=1}^{10} \alpha_k = 10, 0.5\right)$$

If we apply the linear utility  $u(w) = w$  over total rewards, the expected utility associated with alternative  $A$  is

$$E[u \circ Z_A] = \sum_{k=1}^{10} \alpha_k \cdot \beta = 10 \cdot 0.5 = 5.$$

## 2. Compound renewal process rewards

Let  $N(t)$  be a reward counting process and  $\tau_k = T_k - T_{k-1}$  for  $k \geq 2$ ,  $\tau_1 = T_1$  be the *i.i.d.* positive inter-occurrence times.  $N(t)$  provides the number of rewards up to time  $t$ . Set  $\tau_k$  has a distribution  $F$  for  $k = 1, 2, \dots$ . Then,  $\{X_k, \tau_k\}_{k \geq 1}$  is mutually independent. Now, we are interested in the NPV of such rewards,  $Z(t)$ .

$$Z(t) = \sum_{k=1}^{N(t)} e^{-\delta \cdot T_k} X_k$$

With the exponential discount function,  $Z(t)$  implies the aggregate discount total value earned by time  $t$ . Then, the expected utility has a form of

$$E[u \circ Z(t)] = C \int_0^t (u \circ z)(v) dm(v)$$

where  $m(v)$  is a renewal function associated with  $F$  and  $C$  is a constant.  $C$  is determined by the utility function used.

Now we introduce the expected utility with utility function  $u(w) = w$  as an example of the above general form. First, we obtain the expected utility of  $Z(t)$  by using its moment generating function (*mgf*) and, at last, derive the expected utility of limiting  $Z(t)$ . Let  $M_{Z(t)}$  be a *mgf* of  $Z(t)$ . Then, on the analogy of *mgf* by Ghislain

[23],  $M_{Z(t)}(s)$  becomes

$$M_{Z(t)}(s) = \int_t^\infty dF(v) + \int_0^t M_X(s \cdot e^{-\delta v}) M_{Z(t-v)}(s \cdot e^{-\delta v}) dF(v)$$

where  $t > 0$  and  $\delta \geq 0$ . Now let

$$H_\delta(t) = \int_0^t e^{-\delta v} dF(v)$$

then

$$\begin{aligned} H_\delta^{*k}(t) &= \int_0^t e^{-\delta v} dF^{*k}(v) \\ \sum_{k=1}^\infty H_\delta^{*k}(t) &= \int_0^t e^{-\delta v} dm(v) \end{aligned}$$

where  $m(t) = EN(t) = \sum_{k=1}^\infty F^{*k}(t)$ ,<sup>4</sup> a renewal function associated with  $F$  and  $k \geq 0$ . By taking the  $n$ th order derivative at  $s$  and evaluating it at  $s = 0$ , we have the moments

$$\begin{aligned} M_{Z(t)}^{(n)}(0) &= \sum_{k=0}^{n-1} \binom{n}{k} \mu_{n-k} M_{Z(\cdot)}^{(k)}(0) * \sum_{i=1}^\infty H_{n\delta}^{*i}(t) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} \mu_{n-k} \int_0^t e^{-n\delta v} M_{Z(t-v)}^{(k)}(0) dm(v). \end{aligned}$$

Then, the expected utility of the NPV  $Z(t)$  is

$$E[Z(t)] = M_{Z(t)}^{(1)}(0) = \mu_1 \int_0^t e^{-\delta v} dm(v). \quad (3.3)$$

This expected utility shows that  $C$  becomes  $\mu_1 = E[X_1]$  when the utility function is linear (i.e.,  $u(w) = w$ ).

Let  $Q(v) = e^{\delta(v-t)}$  for  $\delta > 0$  and  $t \geq v$ . Then for any  $t$ ,  $Q(v)$  is nonnegative, nonincreasing, and  $\int_0^\infty Q(v) dv < \infty$ . So  $Q(v)$  is *directly Riemann integrable*. Thus,

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<sup>4</sup>  $F^{*k}$  is the  $k$ -fold convolution of  $F$  with itself.



by the *Key Renewal Theorem*,

$$\begin{aligned}\lim_{t \rightarrow \infty} E[Z(t)] &= \frac{\mu_1}{E[\tau_1]} \int_0^\infty e^{-\delta v} dv \\ &= \frac{\mu_1}{E[\tau_1]} \cdot \frac{1}{\delta}.\end{aligned}\tag{3.4}$$

### A wager alternative example with compound renewal process rewards

Suppose that we have a sequence of rewards that are independently and identically occurring with an exponentially distributed inter-occurrence time having mean 0.3 when alternative  $B$  is chosen. Suppose that each reward after a \$15 million initial investment is anticipated to be normally distributed with mean \$0.35 million. We obtain the expected utility of alternative  $B$ , of which rewards form the compounded renewal process.

Let  $X_k$ ,  $k = 1, \dots$  be normal random variables with mean 0.35, which represent the  $k$ th reward amount and  $T_k$ ,  $k = 1, \dots$  be *i.i.d.* random variables for occurrence of the  $k$ th reward. And set  $\tau_k = T_k - T_{k-1}$  for  $k \geq 2$ ,  $\tau_1 = T_1$ , which implies *i.i.d.* inter-occurrence time. Then,  $\{X_k, \tau_k\}_{k \geq 1}$  is mutually independent. In this example, each  $\tau_k$  is exponentially distributed with mean 0.3. Let  $N(t)$  be a reward counting process by time  $t$ . Then, with  $m(v) = EN(v) = \frac{10}{3}v$ , the expected utility for the given rewards process until time  $t$  is, by the formulation (3.3),

$$\begin{aligned}E[u \circ Z_B(t)] &= -15 + 0.35 \int_0^t e^{-0.05 \cdot v} dm(v) \\ &= -15 + \frac{0.35}{0.3} \int_0^t e^{-0.05 \cdot v} dv \\ &= -15 + \frac{0.35}{0.3} \cdot \frac{1}{0.05} (1 - e^{-0.05 \cdot t})\end{aligned}$$

where 0.05 is used for the discount rate  $\delta$  as well as the utility function  $u(w) = w$ .

Moreover, from the formulation (3.4), the expected utility for limiting total rewards of alternative  $B$  is

$$\lim_{t \rightarrow \infty} E[Z_B(t)] = -15 + \frac{0.35}{0.3} \cdot \frac{1}{0.05} = 8.333.$$

The obtained expected utility for the limiting rewards can be verified by taking  $t \rightarrow \infty$  at the above  $E[u \circ Z_B(t)]$ .

### 3. Irreducible positive recurrent Markov chain rewards

Let  $\{X_k\}_{n \geq k \geq 0}$  be a sequence of rewards which form an irreducible positive recurrent Markov chain with invariant probability distribution  $\pi$ . If we define a discount function as  $g_k : \mathbf{X} \rightarrow \mathbb{R}$  such that  $g_k(X_k) = e^{-\delta \cdot k} X_k$ ,  $\bar{g}_n = \frac{1}{n} \sum_{k=1}^n g_k(X_k)$  is the average of the present value of rewards earned by the  $n$ th occurrence. Then, the NPV of the total rewards  $Z$  can be expressed by

$$Z = n\bar{g}_n = \sum_{k=1}^n e^{-\delta \cdot k} X_k.$$

Suppose  $E_\pi(g) = \int_{\mathbf{X}} g(x)\pi(dx)$  such that  $\lim_{n \rightarrow \infty} E(\bar{g}_n) = E_\pi(g)$  with a bounded function  $g : \mathbf{X} \rightarrow \mathbb{R}$ . If  $E_\pi(|g|) < \infty$ ,  $\bar{g}_n$  converges to  $E_\pi(g)$  as  $n \rightarrow \infty$  with probability 1 by *Strong Law of Large Numbers (SLLN)*. If the given Markov chain is simulated,  $\bar{g}_n$  becomes a natural estimate of  $E_\pi(g)$ . Let the variance  $\sigma_g^2$  be

$$\sigma_g^2 = \text{Var}(g(X_0)) + 2 \sum_{i=1}^{\infty} \text{Cov}(g(X_0), g(X_i)). \quad (3.5)$$

Then, for  $0 \leq \sigma_g^2 < \infty$ , the *Central Limit Theorem (CLT)* provides

$$\sqrt{n}(\bar{g}_n - E_\pi(g)) \longrightarrow N(0, \sigma_g^2) \quad \text{in distribution as } n \rightarrow \infty.$$

Thus, if we have an irreducible positive recurrent Markov chain rewards  $\{X_n\}_{n \geq 0}$  with initial stationary distribution  $\pi$  and  $\sigma_g^2$  such that  $0 \leq \sigma_g^2 < \infty$ , for large  $n$ , the NPV of the total rewards  $Z$  can be approximated by a normal random variable

$$Z = \sum_{k=1}^n e^{-\delta \cdot k} X_k \sim N(nE_\pi(g), n\sigma_g^2) \quad \text{in distribution.}$$

This estimates the expected utility for an alternative having irreducible positive recurrent Markov chain rewards as follows:

$$E[u \circ Z] \approx \int_{\mathbb{R}} (u \circ z) \psi(dz)$$

where  $\psi$  is a normal distribution function with mean  $nE_\pi(g)$  and variance  $n\sigma_g^2$ .

Other than the estimation of  $E_\pi(g)$ , finding the proper  $\sigma_g^2$  is challenging and requires specialized techniques. With condition  $\sigma_g^2 < \infty$  in the Markov chain CLT, the covariance in (3.5) must converge into 0 as  $n \rightarrow \infty$ . Then, we can express  $\sigma_g^2$  as the limiting variance

$$\sigma_g^2 = \lim_{n \rightarrow \infty} \text{Var}\left(\frac{1}{n} \sum_{i=1}^n g(X_i)\right).$$

Hastings [24], Fishman [25], and Shruben [26] investigated *standard time series methods* by which the limiting variance can be estimated from the Markov chain itself. Estimation using these methods has been proposed mostly in the operations research literature [27]. Other methods, such as *window estimators* [28] and the Monte Carlo Markov Chain (MCMC) algorithm [29] are discussed by Geyer [30]. For the mean, we develop an algorithm that generates samples from a given Markov chain such that its stationary distribution is precisely our distribution of interest  $\pi$ . Then, the mean can be approximated from  $\bar{g}_n$  using generated samples since  $\bar{g}_n$  converges to  $E_\pi(g)$  as

$n \rightarrow \infty$ . We evaluate the estimator for  $nE_\pi(g)$

$$n \cdot E_\pi[g(X)] \approx \sum_{k=1}^n e^{-\delta \cdot k} X_k$$

by using samples  $S_j$  from the following algorithm for  $X_k$ .

*Metropolis-Hastings algorithm*<sup>5</sup> for Markov chain rewards

Set  $j = 1$ ;

Initialize  $S_j$ ;

Repeat unless the termination condition is satisfied {

    Sample  $Y$  from proposal distribution  $\phi(\cdot|S_j)$ ;

    Sample a random variable  $U(0, 1)$ ;

    Let

$$\alpha = \min\left\{1, \frac{\pi(Y)\phi(S_j|Y)}{\pi(S_j)\phi(Y|S_j)}\right\}$$

    If  $U \leq \alpha$ , then  $S_{j+1} \leftarrow Y$ , else  $S_{j+1} \leftarrow S_j$ ;

    Increment  $j$ ;

}

This *Markov chain Monte Carlo (MCMC)* algorithm generates a sequence of samples  $\{S_j\}_{j \geq 1}$  from an irreducible positive recurrent Markov chain having invariant distribution  $\pi$ . The convergence rate for the stationary distribution highly depends on the relationship between the proposal and stationary distributions. Conventionally, the samples are used for estimation after a sufficiently long *burn-in* period of  $m$  samples are removed. In other words,  $\{S_j\}_{j \geq m+1}$  is used for  $\{X_k\}_{k \geq 1}$ . Proposal distribution  $\phi(\cdot|\cdot)$  can have any form of distribution while the stationary distribution

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<sup>5</sup> A MCMC algorithm widely used in simulating complex, nonstandard multivariate distributions [31, 32, 33].

is the invariant distribution of chain  $\pi$ . The number of samples, i.e.,  $m + n$ , can be a termination condition for the above algorithm.

### **A wager alternative example with irreducible positive recurrent Markov chain rewards**

Let  $X_k$ ,  $k = 1, \dots$  be a sequence of rewards by the selection of alternative C, which forms an irreducible positive recurrent Markov chain with a  $N(0.2, 0.15^2)$  invariant distribution. Suppose that the initial investment is \$9 million and that the projection period is 120. Then, the NPV of total rewards  $Z_C$  of alternative C is

$$Z_C = -9 + \sum_{k=1}^{120} e^{-\delta \cdot k} X_k$$

$Z_C$  can be estimated with a normal random variable having mean  $-9 + 120 \cdot E_\pi(g)$  and variance  $120 \cdot \sigma_g^2$ . We simulate a given Markov chain using the proposed Metropolis-Hastings algorithm to approximate the expected utility. Linear  $u(w) = w$  and 0.0035 are used for the utility function and the discount rate, respectively. Figure 2 shows 1000 samples from this chain with a proposal distribution  $N(\cdot, 0.05^2)$ . The initial value  $S_1 = 2$  is used for verifying the convergence of samples to stationary distribution  $N(0.2, 0.15^2)$ . For the following estimator, 120 samples are applied for  $\{X_k\}_{1 \leq k \leq 120}$  after discarding the first 150 samples as a burn-in. The experiment is conducted on MATLAB.

$$E[u \circ Z_C] \approx 120 \cdot E_\pi(g) \approx -9 + \sum_{k=1}^{120} e^{-0.0035 \cdot k} X_k = 9.7376$$

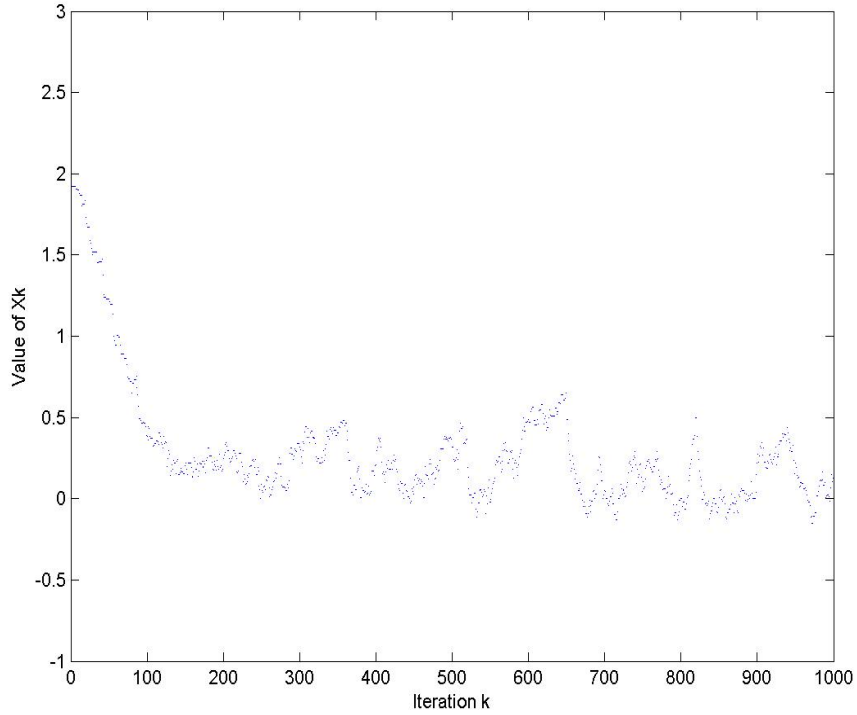


Fig. 2. Sample paths from a Markov chain having a  $N(0.2, 0.15^2)$  invariant distribution.

#### 4. Zero mean martingale rewards

Let  $\{X_n\}_{n \geq 0}$  be a zero-mean, square-integrable martingale that represents a sequence of present values of rewards after the selection of an alternative. Or we can consider  $\{X_n\}_{n \geq 0}$  as the martingale transformed from the martingale rewards which are not present values. The transformation is available since the discount function is bounded and can be prescribed with martingale differences. The transformed sequence again forms a martingale based on the *martingale transform theorem*.<sup>6</sup> Hence, the transform results in the zero-mean, square-integrable martingale. Each variable is adapted to

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<sup>6</sup> This is one of the major results in martingale theory, which is used to model certain betting strategies or gambling systems [34].

the natural filtration  $\{\mathcal{F}_n\}_{n \geq 0}$ , where  $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$ . Let  $\{\xi_n\}_{n \geq 1}$  be the corresponding *martingale differences*

$$\xi_1 = X_1 - X_0, \quad \xi_2 = X_2 - X_1, \dots$$

It is obvious that  $\{\xi_k : 1 \leq k \leq n\} \in \mathcal{F}_n$ . The martingale differences are not necessarily independent but satisfy

$$E[\xi_n | \mathcal{F}_{n-1}] = E[X_n - X_{n-1} | \mathcal{F}_{n-1}] = 0$$

for  $n \geq 1$  and orthogonality

$$E[\xi_k \xi_n] = E[\xi_k E[\xi_n | \mathcal{F}_{n-1}]] = 0$$

for  $k \leq n$ . Although the orthogonality does not imply independence, it is sufficiently close that we might utilize the central limit theorem for the martingale differences [35]. Suppose that the *martingale array* is  $\{X_{n,k}, \mathcal{F}_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$ . The conditional variance  $V_{n,k_n}^2$  and the square variance  $U_{n,k_n}^2$  of  $X_{n,k}$  can be defined with difference  $\xi_{n,k}$  by

$$\begin{aligned} V_{n,k_n}^2 &= \sum_{k=1}^{k(n)} E[\xi_{n,k}^2 | \mathcal{F}_{n,k-1}], \\ U_{n,k_n}^2 &= \sum_{k=1}^{k(n)} \xi_{n,k}^2. \end{aligned}$$

Then the *Martingale central limit theorem* states asymptotical normality and independence (or *mixing*) of random variable  $\frac{X_{n,k(n)}}{U_{n,k_n}}$ .

### ***Martingale Central Limit Theorem***

Let  $\{X_{n,k}, \mathcal{F}_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$  be a zero-mean, square integrable martingale array with differences  $\xi_{n,k}$ , and let  $\eta^2$  be an a.s. finite random

variable such that  $P\{\eta^2 > 0\} = 1$ . Suppose

$$\max_k |\xi_{n,k}| \longrightarrow 0 \quad \text{in probability,} \quad (3.6)$$

$$E[\max_k \xi_{n,k}^2] \text{ is bounded in } n, \quad (3.7)$$

$$\sum_k \xi_{n,k}^2 \longrightarrow \eta^2 \quad \text{in probability,} \quad (3.8)$$

$$\mathcal{F}_{n,k} \supseteq \mathcal{F}_{n-1,k} \quad \text{for } 1 \leq k \leq k_n, n \geq 1. \quad (3.9)$$

Then

$$\frac{X_{n,k_n}}{U_{n,k_n}} \longrightarrow N(0, 1) \quad \text{in distribution.}$$

Although negligibility conditions (3.6) and (3.7) are replaced by the *conditional Linderberg condition*, for all  $\varepsilon > 0$ ,

$$\sum_{k=1}^{k_n} E[\xi_{n,k}^2 \cdot I_{|\xi_{n,k}| > \varepsilon} | \mathcal{F}_{n,k-1}] \longrightarrow 0 \quad \text{in probability} \quad (3.10)$$

and (3.8) is replaced by an analogous condition on the conditional variance:

$$V_{n,k_n}^2 = \sum_{k=1}^{k_n} E[\xi_{n,k}^2 | \mathcal{F}_{n,k-1}] \longrightarrow \eta^2 \quad \text{in probability} \quad (3.11)$$

with the measurability condition on  $\eta^2$ , the conclusion of the martingale CLT remains true [36, 37]. Since the *tight*<sup>7</sup> conditional variance  $V_{n,k_n}^2$  can be approximated by the squared variance  $U_{n,k_n}^2$ , we also have an equivalent result [38]

$$\frac{\sum_{k=1}^{k_n} \xi_{n,k}}{\eta} \longrightarrow N(0, 1) \quad \text{in distribution.}$$

The result implies that a martingale  $\{X_n\}_{n \geq 0}$  has an asymptotic normality with zero mean and  $\eta^2$  variance satisfying uniform integrability of  $V_{n,k_n}^2$ , measurability of  $\eta$ , and conditions (3.10) and (3.11). Suppose that  $S_n$  is the sum of martingales such

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<sup>7</sup> In other words, *uniformly integrable*.



that

$$S_n = \sum_{i=1}^n X_i = \sum_{i=1}^n \sum_{k=1}^i \xi_{i,k} = \sum_{i=1}^n (n-i+1) \xi_{n,i}.$$

We now induce the distribution of  $S_n$  using the asymptotic property from the martingale CLT. If  $\{\xi_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$  is a martingale difference satisfying condition (3.10) and (3.11),  $Y_{n,k} = \frac{n-k+1}{n} \xi_{n,k}$  for  $1 \leq k \leq k_n$  has a conditionally zero-mean and orthogonality property as follows,

$$E[Y_{n,k} | \mathcal{F}_{n,k-1}] = \frac{n-i+1}{n} E[\xi_{n,k} | \mathcal{F}_{n,k-1}] = 0,$$

$$E[Y_{n,k} Y_{n,m}] = \frac{(n-k+1)(n-m+1)}{n^2} E[\xi_{n,k} \xi_{n,m}] = 0 \quad \text{for } k \leq m.$$

The process  $Y_{n,k}$  obviously satisfies conditions (3.10) and (3.11) since  $|\frac{n-k+1}{n}| \leq 1$  for  $1 \leq k \leq n$ . Thus, with measurable  $\eta$ , we can state

$$\sum_{k=1}^{k_n} \frac{n-k+1}{n} \xi_{n,k} \longrightarrow N(0, \eta^2) \quad \text{in distribution.}$$

Therefore, if we have the present values of the zero mean martingale rewards  $\{X_n\}_{n \geq 0}$  satisfying conditions (3.10) and (3.11), the NPV of total rewards  $Z$  can be approximated by a normal random variable as follows:

$$Z = \sum_{k=1}^n X_k = \sum_{k=1}^{k_n} (n-k+1) \xi_{n,k} \sim N(0, n^2 \eta^2) \quad \text{in distribution} \quad (3.12)$$

where  $\{\xi_{n,k}, \mathcal{F}_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$  is a corresponding martingale difference adapted to  $\{\mathcal{F}_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$ , and  $\eta$  is an almost surely finite measurable random variable such that  $V_{n,k_n}^2 \rightarrow \eta^2$  in probability. Then the expected utility of the NPV of total rewards can be approximated as

$$E[u \circ Z] \approx \int_{\mathbb{R}} (u \circ z) \psi(dz) \quad (3.13)$$

where  $\psi$  is a normal distribution function with zero mean and  $n^2 \eta^2$  variance. The

above formulation has various applications in that every well-behaved, real-valued stochastic process can be represented as the sum of a martingale, predictable process by Doob's decomposition, and the representation is unique [39].

### A wager alternative example with zero-mean martingale rewards

Suppose that the present worth of rewards from a selection of alternative  $D$  can be expressed as a square-integrable martingale with mean 0.012 and conditional variance 0.0001. Suppose that the alternative  $D$  has 120 rewards which form the above martingale satisfying conditions (3.10) and (3.11). Let  $\{X_k\}_{k \geq 1}$  be a sequence of present values of zero mean, square integrable martingale rewards from alternative  $D$ , adapted to the natural filtration  $\{\mathcal{F}_n : n \geq 1\}$ , where  $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ . Then from the formulation (12), the total rewards  $Z_D$  is

$$Z_D = -0.012 \cdot 120 + \sum_{i=1}^{120} X_i \sim N(-1.44, 120^2 \cdot 0.0001).$$

If we employ a utility function  $u(w) = a + be^{\rho w}$  with  $a = 20$ ,  $b = -20$ , and  $\rho = 0.004$ , the expected utility of the given martingale rewards associated with alternative  $D$  is, from formulation (13),

$$E[u \circ Z_D] \approx 20 \left( 1 - \int_{\mathbb{R}} e^{-0.004 \cdot z_D} \psi(dz_D) \right) = 7.976$$

where  $\psi(z_D)$  is a normal distribution function with mean  $-1.44$  and variance  $1.44$ . For linear function  $u(w) = w$ ,  $E[u \circ Z_D]$  is estimated as  $-1.44$ .

## 5. Stationary ergodic process rewards

Let  $\{X_n\}_{n \geq 1}$  form an ergodic and stationary process that represents the present values of rewards such that  $\sigma^2 = E[X_1^2] < \infty$ ,  $E[X_{n+1}|\mathcal{F}_n] = 0$  with  $\mathcal{F}_n = \sigma\{X_j : j \leq n\}$ .

By the *CLT for stationary ergodic process*,

$$\frac{\sum_{i=1}^n X_i}{\sqrt{n}} \longrightarrow N(0, \sigma^2) \quad \text{in distribution.}$$

The NPV of the stationary ergodic process rewards up to time  $n$  can be estimated by the normal random variable

$$Z = \sum_{i=1}^n X_i \sim N(0, n\sigma^2).$$

Then the expected utility of the given stationary ergodic process rewards can be approximated by

$$E[u \circ Z] \approx \int_{\mathbb{R}} (u \circ z) \psi(dz)$$

where  $\psi$  is a normal distribution function with zero mean and  $n\sigma^2$  variance.

### **A wager alternative example with stationary ergodic process rewards**

Suppose that  $\{X_n\}_{120 \geq n \geq 1}$  is a sequence of rewards from a selection of alternative  $E$ . Suppose that the rewards are estimated to be a stationary ergodic process having mean 5 and variance  $\sigma^2 = 0.015$ . Then the NPV of such rewards can be estimated from a normal distribution with mean  $0.042$  and variance  $120 \cdot 0.015 = 1.8$

$$Z_E = 0.042 \cdot 120 + \sum_{i=1}^{120} X_i \sim N(5.04, 1.8).$$

In addition, the expected utility of the given rewards is approximated by

$$E[u \circ Z_E] \approx \int_{\mathbb{R}} (u \circ z_E) \psi(dz_E)$$

where  $\psi(z_E)$  is a normal distribution function with mean 5.04 and variance 1.8. With the utility function  $u(w) = a + be^{\rho w}$ ,  $a = 20$ ,  $b = -20$  and  $\rho = 0.004$ , the expected utility becomes 14.892 while it is 5.04 with a utility function  $u(w) = w$ .

#### D. Preference Ordering of Special Wager Alternatives

The expected utility is the quantitative value for a wager alternative based on the gambler's attitude toward total rewards of the alternative. Thus, selecting the most preferred alternative from among marked point process wager alternatives appeals finding the maximum value among the expected utilities for such wager alternatives. Let us consider two expected utilities,  $E_\alpha$  and  $E_\beta$ , induced by wager alternatives  $\alpha$  and  $\beta$ , respectively. With a strictly increasing utility function  $u$ , the statement that “ $E_\alpha$  is greater than  $E_\beta$ ” implies that  $\alpha$  is preferred over  $\beta$  or  $\alpha \succ \beta$ . So, we have the following *equivalence relation* [10]

$$E_\alpha > E_\beta \quad \Leftrightarrow \quad \alpha \succ \beta.$$

In the previous section, we investigated the risks and the expected utilities for some wager alternatives whose rewards form one of special marked point processes; sequence of i.i.d. random variables in discrete time, compound renewal process, irreducible positive recurrent Markov Chain, zero-mean martingale, and zero-mean square integrable stationary ergodic process. With the expected utilities obtained in examples (i.e.,  $E_A = 5$ ,  $E_B = 8.333$ ,  $E_C = 9.7376$ ,  $E_D = -1.44$ , and  $E_E = 5.04$ ), we can order the preferences of the risks associated with the special wager alternatives as follows:

$$E_C > E_B > E_E > E_A > E_D \quad \Leftrightarrow \quad C \succ B \succ E \succ A \succ D.$$

where the utility function  $u(w) = w$  is used. In the above relation, alternative  $C$  is the most favorable while alternative  $D$  is the least attractive for a gambler with the utility function used.

In this chapter, we introduced the risks associated with wager alternatives whose rewards can be represented as marked point processes. Distribution function for NPV of a reward process is used for risk of these wager alternatives. The EUT and its supporting axioms are illustrated from the wagering point of view. Then, the expected utilities from the EUT are available for comparing the values of wager alternatives based on gambler's preference. The wager alternative having the most favorable risk is then the one associated with the greatest expected utility. However, the more number of rewards are appeared for a wager alternative, the harder we analytically obtain the corresponding expected utility of the alternative. Nonetheless, there are some cases in which the risk is easily accessible even though the alternative retains a large number of rewards. When the rewards of a wager alternative are independently and identically distributed, the expected utility can be computed through the joint distribution function of rewards. For a wager alternative having a large number of compound renewal process rewards, we introduced how to obtain the expected utility using the *mgf* of the NPV of rewards and the Key Renewal Theorem to find the limiting expected utility. For wager alternatives whose rewards form irreducible positive recurrent Markov Chain, zero mean martingale and stationary ergodic process, we developed the estimators of the expected utilities with the foundation of the CLT. For the Markov Chain case, a MCMC algorithm to approximate the mean of the limiting distribution is constructed for practical usage. For these special marked point process rewards, calculation of the expected utilities becomes much simpler, equipped with the presented expression of the expected utilities. Then, we can easily achieve preference ordering among the wager alternatives with special marked point process rewards as shown in the example of the last section.

The marked point process wager alternatives discussed in this chapter have an unique

risk for each alternative. However, it is often impractical to characterize the unique risk for a wager alternative. In the next chapter, we discuss such circumstance and develop the computational framework available for identifying the expected utilities non-unique risk of a wager alternative map into. The principles that provide the foundation of weak preference ordering among such wager alternatives are also presented.

## CHAPTER IV

PARTITIONING WAGER ALTERNATIVES UNDER NON-UNIQUE RISK  
CHARACTERIZATION

In the previous chapter, we considered special wager alternatives and the separation of preferences by risk based on the EUT; this approach requires that the risk of each alternative be available. However, in the absence of either rich historical data or confidence about the future rewards, it is generally unrealistic to identify a unique distribution function for the risk of a wager alternative. Instead, we often capture uncertainty only on some subset of reward events. Such incomplete information about rewards may result in a set of distribution functions to which the risk for the given wager alternative must belong. With this non-unique risk characterization of the wager alternative, the approaches that are described in the previous chapter for identifying the risks and choosing the most favorable alternative are not sufficient. Consequently, new mathematical models and theoretical foundations for the preference ordering of such wager alternatives are needed.

Given a unique utility function, the EUT provides a measurable mapping between a distribution function and a real number. If a collection of all of the distribution functions characterizing the uncertainty of an alternative is assumed as a domain of a *functional*<sup>1</sup>, we can extend the EUT with this domain so that a set of expected utilities is generated. This set can then be expressed as an interval of  $\mathbb{R}$  where the lower or the upper bounds consist of the minimum or the maximum expected utilities, respectively, which implies that the interval contains the expected utilities that the risk of the wager alternative must map into. We introduce a computational framework

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<sup>1</sup> It is a function that takes functions as its argument.

for identifying these bounds from our understanding of the rewards. This framework utilizes the information on rewards to construct a set of feasible risks associated with the alternative. This information is classified into five categories, based on the characteristics of the data. For each type of data, we present the mathematical formulation applicable for placing a restriction on the set of risks. By combining all of these formulations, a set of feasible risks for the wager alternative can be constructed. Hence, a set of expected utilities associated with wager alternatives having non-unique risks can be identified through this framework. Since there might be a non-empty intersection among some sets of expected utilities, the choice of a preferred alternative is a complicated issue. An interval graph representation is applied to describe the relationship among the sets of expected utilities. Utilizing this expression, we develop some theorems and a procedure for possible preference ordering of such wager alternatives. At the end of this chapter, we illustrate a situation where the selection of the alternative with the most favorable risk is impracticable and, in another case, we show how to partition the alternatives under a non-unique risk characterization.

#### A. Mathematical Formulation

For the wager alternatives only certain reward event probabilities are available, since unique risk characterization is often inadequate, we develop a mathematical formulation that allows us to identify the non-unique risks and their corresponding expected utilities. Because the set of expected utilities might be bounded, the interval of  $\mathbb{R}$  can be used for this set when the lower and the upper bounds of the interval represent the minimum and the maximum expected utilities in the set.<sup>2</sup> The formulation below

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<sup>2</sup> Or we might regard the interval as the smallest set that covers all of the expected utilities that the risks of such wager alternatives are mapped onto using the EUT



identifies the risks and the expected utilities by seeking these lower and upper bounds on the interval.

Let  $\alpha$  be a wager alternative only some reward event probabilities are available and let  $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$  be a probability space on which all of the random elements are defined. Let  $\{(X_n^\alpha, T_n^\alpha) : n \in \mathbb{Z}_+\}$  be a sequence of the marked point process rewards for alternative  $\alpha$ , in which  $X_k^\alpha$  represents a  $k$ th reward amount while  $T_k^\alpha$  represents  $k$ th reward occurrence time. None of these random variables have to be independent. Let  $\mathcal{A}_\alpha$  be a set of distribution functions consistent with the given reward information for alternative  $\alpha$ .<sup>3</sup> This becomes the set of feasible risks. Then, as noted previously, the expected utilities from the EUT lie within an interval  $I_\alpha$  such that  $\int_{\mathbf{X}} u(z) dF_\alpha(z) \in I_\alpha$  for all  $F_\alpha \in \mathcal{A}_\alpha \subset D(\mathbf{X})$ . The lower and upper bounds of the interval  $I_\alpha = [l_\alpha, u_\alpha]$  can then be obtained by

$$l_\alpha = \min_{F_\alpha \in \mathcal{A}_\alpha} \int_{\mathbf{X}} u(z) dF_\alpha(z)$$

$$u_\alpha = \max_{F_\alpha \in \mathcal{A}_\alpha} \int_{\mathbf{X}} u(z) dF_\alpha(z).$$

The above formulations seek two  $F_\alpha$ s that minimize and maximize the expected utilities. The risk of alternative  $\alpha$  must map into the interval bounded by these expected utilities. With these formulations, the value of an alternative with non-unique risk characteristics can be measured without counting all of the distribution functions  $F_\alpha \in \mathcal{A}_\alpha$ .

Now, we rewrite  $\mathcal{A}_\alpha$  by employing the given reward event probability data in a joint distribution function of the rewards. This information places restrictions on the joint distribution function and finally establishes the set of feasible risks. Let  $G(\underline{y})$ ,  $\underline{y} = (y_1, \dots, y_{2n})$ , be a joint distribution function on the rewards of wager alternative

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<sup>3</sup> Construction of  $\mathcal{A}_\alpha$  is described in the following section.

$\mathcal{A}_\alpha$ , such that  $G(\underline{y}) = P\{X_1 \leq y_1, \dots, X_n \leq y_n, T_1 \leq y_{n+1}, \dots, T_n \leq y_{2n}\}$  as shown in (3.2). Define  $\mathcal{A}'_\alpha$  as a set of distribution functions such that  $G_\alpha \in \mathcal{A}'_\alpha$  corresponds to  $\mathcal{A}_\alpha$ . Let  $G(\underline{c}_i)$  be the distribution function with a vector of constraints  $i$  for the alternative  $\alpha$ . Then,  $G(\underline{c}_i)$  is constrained by probability measures on reward event  $i$ . The information obtained from our understanding of the reward probability can be written in the canonical form  $\varphi_\alpha(G(\underline{c}_1), \dots, G(\underline{c}_k)) \leq 0$  where  $\varphi_\alpha : \mathbb{R}^k \mapsto \mathbb{R}$  is a functional representation of the reward probabilities for alternative  $\alpha$  in terms of joint distribution  $G$ . By representing  $\mathcal{A}'_\alpha$  with the canonical form, the lower and upper bounds of the interval for alternative  $\alpha$  can be obtained from

$$\begin{aligned} l_\alpha &= \min_{G \in \mathcal{A}'_\alpha} \int_{\mathbb{R}^n \times \mathbb{R}_+^n} (u \circ z)(\underline{y}) dG(\underline{y}) \\ \text{s.t. } \mathcal{A}'_\alpha &= \{G : \varphi_\alpha(G(\underline{c}_1), \dots, G(\underline{c}_k)) \leq 0\} \end{aligned} \quad (4.1)$$

$$\begin{aligned} u_\alpha &= \max_{G \in \mathcal{A}'_\alpha} \int_{\mathbb{R}^n \times \mathbb{R}_+^n} (u \circ z)(\underline{y}) dG(\underline{y}) \\ \text{s.t. } \mathcal{A}'_\alpha &= \{G : \varphi_\alpha(G(\underline{c}_1), \dots, G(\underline{c}_k)) \leq 0\} \end{aligned} \quad (4.2)$$

where  $G(\underline{y}) = P\{X_1 \leq y_1, \dots, X_n \leq y_n, T_1 \leq y_{n+1}, \dots, T_n \leq y_{2n}\}$ ,  $\underline{y} = (y_1, \dots, y_{2n})$  for random variables that represent the amount and the occurrence time of rewards.

The formulations (4.1) and (4.2) are in a format congruent with *constrained variational calculus* [40]. They provide the interval  $I_\alpha = [l_\alpha, u_\alpha]$  where the expected utilities associated with the risk of alternative  $\alpha$  must lie. Using such formulations, the identification of every expected utility is unnecessary to attain our objective of ordering the wager alternatives by risk preference. In the above formulations, the information about reward event probabilities plays the role of constraints. The characterization of  $\mathcal{A}'_\alpha$  is introduced in the next section.

## B. Construction of Feasible Risk Sets

In this section, we construct the set of feasible risks which is described by a  $\mathcal{A}'_\alpha$  using constrained variational calculus (4.1) and (4.2). Let  $\Omega_\alpha$  be the set of all possible outcomes related to the rewards when alternative  $\alpha$ , which has a non-unique risk characterization, is selected. We define  $\{(X_n^\alpha, T_n^\alpha) : n \in \mathbb{Z}_+\}$  as a sequence of random variables in which  $X_k^\alpha$  and  $T_k^\alpha$  represent the amount of the  $k$ th reward event and its occurrence time for alternative  $\alpha$ . It is a sequence of mapping into  $\mathbb{R} \times \mathbb{R}_+$ . The *natural filtration* for alternative  $\alpha$  is then

$$\mathcal{F}_n^\alpha = \sigma((X_0, T_0), (X_1, T_1), \dots, (X_n, T_n))$$

which is an increasing family of the sub  $\sigma$ -algebra of  $\mathcal{F}_\alpha$ . If probability measure  $P : \mathcal{F}_\alpha \rightarrow [0, 1]$  is assigned to  $(\Omega_\alpha, \mathcal{F}_\alpha)$ , a probability triple  $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$  forms a probability space for wager alternative  $\alpha$ . Then, the risks associated with the wager alternative are represented as the set of distribution functions in this probability space. Here,  $\Omega_\alpha$  is not necessarily countable. If it is a countable set,  $P$  is determined by the probabilities of events that constitute  $\Omega_\alpha$ , and the probability measures are expressed by discrete distribution functions. The set of feasible risks can be identified by imposing conditions on the distribution functions (i.e.,  $\varphi_\alpha(G(\underline{c}_1), \dots, G(\underline{c}_k)) \leq 0$ ). These conditions are based on information about the probabilities of the alternative rewards. An instance of the condition is  $P(\Omega_\alpha) = 1$ . If  $G$  is a joint distribution function on the rewards of wager alternative  $\alpha$  such that  $G(\underline{y}) = P\{X_1 \leq y_1, \dots, X_n \leq y_n, T_1 \leq y_{n+1}, \dots, T_n \leq y_{2n}\}$  and  $G \in \mathcal{A}'_\alpha$ , this necessary condition can be rewritten by

$$\int_{\mathbb{R}^n \times \mathbb{R}_+^n} dG(\underline{y}) = 1 \quad (4.3)$$

We classify other conditions into three categories based on the type of probability information: distributional bounds, ordinal distributions, and independence. The first category includes bounded marginal and bounded conditional distribution functions. We introduce this information and present their corresponding conditions in mathematical forms.

### 1. Bounded marginal distribution

Suppose that the amount of the  $k$ th reward,  $X_k$ , is less than, or equal to,  $y_k$ , or  $\{X_k \leq y_k\}$ . It is equivalent to the set of events satisfying this inequality for the  $k$ th reward amount. If the probability measures are applied to an element of this set, the collection of the probability measures can be rewritten by a marginal joint distribution function,  $P\{X \leq y_k\}$ . We often encounter information that might be a restriction for the probability measures, that is, a bound for the marginal distribution function. Suppose that the collection of probability measures for the events of which the amount of the  $k$ th reward is less than, or equal to,  $y_k$  lies within constants  $c_1 \in [0, 1]$  and  $c_2 \in [0, 1]$  such that  $c_1 \leq c_2$ . It can be expressed by

$$c_1 \leq P\{X_k \leq y_k\} \leq c_2 \quad (4.4)$$

for  $k = 1, \dots, n$ . This condition of a bounded marginal distribution can be constructed from various sources, e.g., historical data, estimates, and it can be used as a constraint in  $\mathcal{A}'_\alpha$ . We can have the same type of condition for the marginal distribution of reward times.

## 2. Bounded conditional distribution

There is another type of events set, from which the probability measures are collectible up to a certain point based on our level of knowledge. Let  $E_j$  be the set that the amount of the  $j$ th reward is less than, or equal to,  $y_j$ , or  $E_j = \{X_j \leq y_j\}$ . We can capture a different set restricted to  $E_j$ , e.g., the amount of the  $k$ th reward is greater than, or equal to,  $y_k$ , given  $E_j$ , or  $\{X_k \geq y_k \mid E_j\}$ . This leads to a conditional distribution when probability measures are applied. Then, the bound information on the conditional distribution can be rewritten as

$$P\{X_k \geq y_k \mid X_j \leq y_j\} \geq c \quad (4.5)$$

for  $j, k = 1, \dots, n$  and a constant  $c \in [0, 1]$ . That is, by rearrangement,

$$P\{X_k \geq y_k, X_j \leq y_j\} - cP\{X_j \leq y_j\} \geq 0.$$

Moreover, we can express this constraint for a disjoint set of atomic events. Let  $E_j$  and  $E_k$  be the sets of atomic events. Then,

$$\begin{aligned} P\{E_j \cap E_k\} - c \cdot P\{E_j\} &= P\{E_j \cap E_k\} - c \cdot \left( P\{E_j \cap E_k\} + P\{E_j \setminus E_k\} \right) \\ &= (1 - c)P\{E_j \cap E_k\} - c \cdot P\{E_j \setminus E_k\} \geq 0 \end{aligned}$$

where  $\{E_j \cap E_k\}$  and  $\{E_j \setminus E_k\}$  are disjoint sets with set intersection  $\cap$  and set difference  $\setminus$ . With  $G(\underline{y})$ , the condition (4.5) becomes

$$(1 - c) \int_{\mathbb{R}^{n-2}} \int_{y_k}^{\infty} \int_{-\infty}^{y_j} \int_{\mathbb{R}_+^n} dG(\underline{y}) - c \int_{\mathbb{R}^{n-2}} \int_{-\infty}^{y_k} \int_{y_j}^{\infty} \int_{\mathbb{R}_+^n} dG(\underline{y}) \geq 0.$$

## 3. Ordinal distributions

One type of data that might have about the rewards of a wager alternative is the ordinal relationship among the probability measures for events. Suppose that we

have two event sets,  $\{X_j \leq y_j\}$  and  $\{X_k \leq y_k\}$ . The ordinal information for the probability measures of these sets might result in

$$P\{X_j \leq y_j\} \leq \varepsilon P\{X_k \leq y_k\} \quad (4.6)$$

with a constant  $\varepsilon$ . Let  $E_j$  and  $E_k$  be sets such that  $E_j = \{X_j \leq y_j\}$  and  $E_k = \{X_k \leq y_k\}$ . Then constraint (4.6) can be rewritten as

$$\begin{aligned} P\{E_j\} - \varepsilon P\{E_k\} &= P\{(E_j \cap E_k) \cup (E_j \setminus E_k)\} - \varepsilon P\{(E_j \cap E_k) \cup (E_k \setminus E_j)\} \\ &= (1 - \varepsilon)P\{(E_j \cap E_k)\} + P\{E_j \setminus E_k\} - \varepsilon P\{E_k \setminus E_j\} \leq 0 \end{aligned}$$

This inequality, expressed with disjoint sets of atomic events, carries the condition as in (4.6). By using a joint distribution function,  $G(\underline{y})$ , it is again

$$\begin{aligned} (1 - \varepsilon) \int_{-\infty}^{y_k} \int_{\mathbb{R}^{n-2}} \int_{-\infty}^{y_j} \int_{\mathbb{R}_+^n} dG(\underline{y}) + \int_{y_k}^{\infty} \int_{\mathbb{R}^{n-2}} \int_{-\infty}^{y_j} \int_{\mathbb{R}_+^n} dG(\underline{y}) \\ - \varepsilon \int_{-\infty}^{y_k} \int_{\mathbb{R}^{n-2}} \int_{y_j}^{\infty} \int_{\mathbb{R}_+^n} dG(\underline{y}) \leq 0. \end{aligned}$$

The ordinal relationship among distributions and the distributional bounds form linear constraints that constitute the set of distribution functions  $\mathcal{A}'_\alpha$  for wager alternative  $\alpha$ . However, the independence characteristic of the events in the following section introduces nonlinearity in  $\mathcal{A}'_\alpha$ .

#### 4. Independence

Let  $E_j = \{X_j \leq y_j\}$  and  $E_k = \{X_k \leq y_k\}$  be two events sets. If the  $\sigma$ -algebra generated by these sets is independent, sets  $E_j$  and  $E_k$  are independent. Moreover, probability measures on the intersection of these sets can be represented by the mul-

tiplication of probability measures of each sets; for instance,

$$P\{X_j \leq y_j, X_k \geq y_k\} = P\{X_j \leq y_j\}P\{X_k \geq y_k\}. \quad (4.7)$$

With sets  $E_j$  and  $E_k$ , we rewrite the equation (4.7) as

$$\begin{aligned} P\{E_j \cap E_k\} - P\{E_j\}P\{E_k\} &= P\{E_j \cap E_k\} - P\{(E_j \cap E_k) \cup (E_j \setminus E_k)\} \\ &\quad \cdot P\{(E_j \cap E_k) \cup (E_k \setminus E_j)\} \\ &= -(P\{E_j \cap E_k\})^2 + P\{E_j \cap E_k\}(1 - P\{E_j \setminus E_k\} \\ &\quad - P\{E_k \setminus E_j\}) + P\{E_j \setminus E_k\}P\{E_k \setminus E_j\} \\ &= 0 \end{aligned}$$

in terms of disjoint sets of atomic events. By using the joint distribution function  $G(\underline{y})$ , it becomes

$$\begin{aligned} &-\left(\int_{y_k}^{\infty} \int_{\mathbb{R}^{n-2}} \int_{-\infty}^{y_j} \int_{\mathbb{R}_+^n} dG(\underline{y})\right)^2 + \left(\int_{y_k}^{\infty} \int_{\mathbb{R}^{n-2}} \int_{-\infty}^{y_j} \int_{\mathbb{R}_+^n} dG(\underline{y})\right) \\ &\quad \cdot \left(1 - \int_{-\infty}^{y_k} \int_{\mathbb{R}^{n-2}} \int_{-\infty}^{y_j} \int_{\mathbb{R}_+^n} dG(\underline{y}) - \int_{y_k}^{\infty} \int_{\mathbb{R}^{n-2}} \int_{y_j}^{\infty} \int_{\mathbb{R}_+^n} dG(\underline{y})\right) \\ &\quad + \left(\int_{-\infty}^{y_k} \int_{\mathbb{R}^{n-2}} \int_{-\infty}^{y_j} \int_{\mathbb{R}_+^n} dG(\underline{y})\right) \cdot \left(\int_{y_k}^{\infty} \int_{\mathbb{R}^{n-2}} \int_{y_j}^{\infty} \int_{\mathbb{R}_+^n} dG(\underline{y})\right) = 0. \end{aligned}$$

As shown above, any independence relationship among events elicits a nonlinear equation, which confers nonlinearity on  $\mathcal{A}'_{\alpha}$ .

Our understanding of the reward probabilities of a wager alternative such as the distributional bounds (4.4), (4.5), ordinal distributions (4.6), and independence (4.7) provides restrictions on  $D(\mathbf{X})$ . These conditions embody  $\mathcal{A}'_{\alpha}$ , a set of distribution functions characterizing a set of feasible risks for wager alternative  $\alpha$ . Thus, by solving optimization formulations (4.1) and (4.2), subject to the four types of conditions discussed so far, we can identify the interval  $I_{\alpha} = [l_{\alpha}, u_{\alpha}]$  for a wager alternative  $\alpha$

having non-unique risk characterization. The set of feasible risks consistent with the known probability information on rewards is then mapped onto interval  $I_\alpha$  or the set of expected utilities. A solution method for the proposed optimization formulation (4.1) and (4.2) will be developed in the next chapter. In the following section, we elucidates two situations: wager alternatives are that either separable or nonseparable based on risk preferences. Principles that support the best alternative selection among wager alternatives with limited reward probability data are also presented.

### C. Preference Ordering among Wager Alternatives under Non-unique Risk Characterization

Suppose that we have a finite set of wager alternatives from which we will select the ones that are most favorable. As defined before, let a distribution function of the NPV of the anticipated rewards from alternative  $\alpha$  be  $G_\alpha \in D(\mathbf{X})$ , where  $D(\mathbf{X})$  is a collection of distribution functions on the support of the NPV of the rewards. Let  $\mathcal{A}'_\alpha \subset D(\mathbf{X})$  be a set of distribution functions that contains the risk of alternative  $\alpha$  such that  $G_\alpha \in \mathcal{A}'_\alpha$ . Define  $l_\alpha$  and  $u_\alpha$  as in (4.1) and (4.2) such that every expected utility mapped from  $\mathcal{A}'_\alpha$  is bounded and within the interval  $I_\alpha = [l_\alpha, u_\alpha]$ . Then each wager alternative can be represented by the expected utilities within the interval or the single expected utility on the real line in case of  $l_\alpha = u_\alpha$ .

For wager alternatives whose expected utilities are completely disjoint on the real line, separating them by risk preferences and selecting ones having the most favorable risks can be easily achieved. However, we cannot separate wager alternatives whose risks are mapped into the same expected utilities. This is the case when the intervals associated with wager alternatives are at least partially overlapped on the real line. In



other words, only disjoint sets of expected utilities make wager alternatives separable. With this reasoning, theoretical establishments are needed that address alternative ordering and the choice of the most favorable among such wager alternatives. We appeal to a lemma first, introduced by Wortman and Park [41], that provides a logical foundation for searching the most preferable alternatives among marked point process wager alternatives. This lemma agrees with Hazelrigg's assertion: "An alternative  $\alpha$  can only be separated from alternative  $\beta$  if all of the outcomes from alternative  $\alpha$  are preferred to the outcomes from alternative  $\beta$ , or vice versa" [42]. They employ simple graph theory for the construction of this lemma.

Let  $G = (V, E)$  be the interval graph where  $V$  and  $E$  represent the set of vertices and edges respectively. Let  $V = \{I_\alpha : \alpha \in \mathcal{T}\}$ , associating a vertex  $v_\alpha \in V$  with each interval  $I_\alpha = [l_\alpha, u_\alpha]$ . This graph is an undirected graph having 1 for  $e_{\alpha\beta} \in E$  if, and only if,  $I_\alpha \cap I_\beta$  is not empty. Let  $\alpha^* = \operatorname{argmax}_{\alpha \in \mathcal{T}} \{l_\alpha\}$ . Then  $I_{\alpha^*}$  is the interval with the greatest lower bound.

***Lemma***

With  $G = (V, E)$  and  $\{I_\alpha; \alpha \in \mathcal{T}\}$  as defined above,  $v_{\alpha^*} \in V$  belongs to a maximal clique and is incident to no other cliques.

***Proof.*** Let  $V_{\alpha^*}$  be the subset of vertices incident to  $v_{\alpha^*}$ , and let  $S$  be the index set of  $V_{\alpha^*}$ . Clearly,  $d = \min_{\gamma \in S} \{u_\gamma\}$  is well defined and  $d > l_{\alpha^*}$ . It follows that  $[l_{\alpha^*}, d] \subset I_\alpha$  for  $\forall \alpha \in S$ . Hence,  $\cap_{\alpha \in S} I_\alpha \neq \phi$ , and by construction of an interval graph, the vertices of  $V_{\alpha^*}$  are mutually incident. That is, the vertices of  $V_{\alpha^*}$  belong to a clique. Since  $v_{\alpha^*}$  is incident to no vertices outside of  $V_{\alpha^*}$ , there can be no clique of  $G$  that properly contains the clique to which  $v_{\alpha^*}$  belongs. Thus,  $V_{\alpha^*}$  is the vertex set of a maximal clique.  $\square$

This lemma describes how preference ordering for wager alternatives can be achieved by ordering cliques composed of intervals associated with alternatives. Application of the lemma for the separation of risk preferences via a family of distribution functions is given by the following corollary.

**Corollary** (Separation of preferences via distribution families)

- (i) Any design alternative with risk belonging to the family of distribution functions  $\mathcal{A}'_\beta$  with  $u_\beta < l_{\alpha^*}$ , is less preferable than any design alternative having risk belonging to the family  $\mathcal{A}'_{\alpha^*}$ .
- (ii) Any design alternative  $\gamma$  for which  $u_\gamma \geq l_{\alpha^*}$  is indistinguishable from the most preferred alternative.

From this corollary, finding the most preferred set of alternatives can be accomplished by questing the maximal clique that includes the greatest lower bound among intervals associated with wager alternatives under non-unique risk characterization. In addition, wager alternatives mapped into the expected utilities bounded by the intervals of this maximal clique are not separable. Based on the theorems, we set up the following procedure in order to determine the most favorable alternative among the marked point process wager alternatives:

- Step 1. For each alternative  $\alpha \in \mathcal{T}$ , characterize a risk with a distribution function  $G_\alpha$  or a set of distribution functions  $\mathcal{A}'_\alpha$ , if appropriate.
- Step 2. Identify either the expected utility  $E_\alpha$  or the interval  $I_\alpha = [l_\alpha, u_\alpha]$  corresponding to the risk for each alternative.
- Step 3. Find the alternative  $\alpha^*$  with the greatest lower bound  $l_{\alpha^*}$ .
- Step 4. Compare the upper bound  $u_\alpha$  of other alternatives with  $l_{\alpha^*}$ .

A risk characterization for a marked point process wager alternative with limited reward probability information was described earlier in this chapter. By solving the constrained calculus of variations (4.1) and (4.2), the interval that contains the set of expected utilities can be identified for each wager alternative. Then, in step 3, the alternative  $\alpha^*$  can be obtained when the lower bounds  $l_\alpha$ s are sorted. When alternative  $\beta \in \mathcal{T}$  has a unique distribution as the risk, we assume  $E_\beta = l_\beta = u_\beta$ . In step 4, we can collect the alternatives associated with the intervals whose upper bounds exceed  $l_{\alpha^*}$ . Then, the alternatives in the final set are the most favorable, and they are not reciprocally separable.

In this chapter, we focused on marked point process wager alternatives for which the information on reward probabilities is not enough to characterize the uncertainty as a unique risk. In order to find the most favorable alternative among such wager alternatives, some difficulties must be overcome. These include how to identify the set of risks that capture the uncertainty of each alternative using the given limited data and how to order the preferences of the alternatives associated with such non-unique risks. We presented mathematical formulations to determine the bounded interval that is consistent with the set of expected utilities for a wager alternative. This formulation utilizes the limited data on reward probabilities for constructing the conditions that any distribution function of the NPV of rewards must satisfy. A set of distribution functions that meet these restrictions then forms the set of feasible risks for the alternative. By applying these formulations, we can identify the set of expected utilities the feasible risks map into in the form of the real interval without exploring every single risk. We introduced a lemma and a corollary to support the choice of the alternative based on risk preference among these wager alternatives. The theorem shows that the most preferred set of alternatives can be obtained by

partitioning the alternatives into cliques and finding the maximal clique that includes the greatest lower bound among the intervals. Finally, a procedure was presented for the practical application of these theorems for selecting the most favorable alternatives among wager alternatives under a non-unique risk characterization.

In the mathematical formulation described in this chapter, the computational burden increases with the size of the reward process. Additionally, only specific types of constrained variational calculus problems can be solved by the methods presented so far. In the next section, we describe an approach applicable for solving given optimization formulations and examine some special cases using this approach.

## CHAPTER V

NUMERICAL APPROACH FOR SEPARATION OF PREFERENCES AMONG  
WAGER ALTERNATIVES

In the previous chapter, a procedure was introduced for finding the marked point process wager alternatives that have the most favorable risks. This procedure employed a constrained variational calculus formulation that includes information on reward probabilities in the form of constraints. This mathematical formulation yields bounds on the expected utilities for a given alternative under non-unique risk characterization, but it is, in practice, non-convex and nonlinear.

It is known that constrained variational calculus formulation can be transformed into variational problems with no constraints that are solvable through the *Euler Lagrange equation* [43, 44, 45]. However, this direct method is typically impractical for realistic problems. First, the Euler Lagrange equation is a partial differential equation that has a solution with a complex structure. Hence, the closed analytical expression of the solution is tractable only in very special circumstances [46]. In addition, there is no guarantee that the globally optimal solution can be obtained in the presence of non-linearity or non-convexity [47]. Hence, the approximation techniques for a solution, i.e., *Numerical Partial Differential Equations (Numerical PDE)* [48], are of growing importance for constrained variational calculus. Numerical approaches such as the Finite Difference Method (FDM) [49], the Finite Element Method (FEM) [50], the Finite Volume Method (FVM) [51], and the Spectral Method (SM) [52] generally classify the regions that the functionals support into finite subspaces in order to apply types of equations other than partial differential equations. However, these approaches often require a high performance computing system and are only applicable for specific formulations. For the constrained variation problems considered in

this research, we must develop an appropriate solution method to identify expected utility bounds.

We present a numerical approach to separate the wager alternatives based on risk preferences. This approach consists of two parts: the construction of standard mathematical programming problems and the application of a solution algorithm. By partitioning events into tiers, the original optimization formulation can be reformulated into a standard programming problem that yields piece-wise constant risks. The decision variables of this problem represent the probabilities of reward amounts and times. A solution method is designed that takes into account our formulation properties and possesses computational efficiency. The method again includes two algorithms; 1) a Markov Chain Monte Carlo (MCMC) algorithm for searching for a better feasible distribution function, and 2) an *Importance sampling* algorithm [53] [54]. This sampling method is embedded in the MCMC algorithm and approximates the expected utility that corresponds to the solution. With this method, the expected utility bounds can be identified for wager alternatives, and we can then separate the alternatives by risk preference. However, we might have a lesser computational burden by utilizing a parallel processing system.

The following section introduces the discretization scheme and the reformulated mathematical programming problems. The solution method for this formulation is presented in section B. Section C is devoted to explaining how to identify expected utility bounds using the discretized formulations and the solution method. The separation of alternatives and the choice of the most favorable alternative for a wager in which the alternative rewards are complicated marked point processes are discussed in section D. This chapter ends with an example that demonstrates the application of the numerical approach for the separating preferences among the marked point process wager alternatives.

### A. Construction of Mathematical Programming Problems

Formulations (4.1) and (4.2) lead to the expected utility bounds for wager alternatives with probability assessments for some subset of events. We employ a discretization scheme for the probability measures on events in order to reformulate (4.1) and (4.2) into standard mathematical programming problems. First, we partition the events into classes and assign variables for the probability measures, where each variable represents the probability measures on the elements of a class. The distribution function is expressed by the sum of variables, and the formulations (4.1) and (4.2) are rewritten using these variables.

Next, we define a tier as a number that represents a class where partitioned events are contained. For computational convenience, this number is generally selected from the class that the tier represents. A set of tiers is then a subset of integers for reward amounts or a subset of non-negative integers for reward inter-occurrence time. Let  $A \subset \mathbb{Z}$  be a set of tiers classified from reward amount events, and  $B \subset \mathbb{Z}_+$  be a set of tiers classified from reward inter-occurrence time events. Define  $\rho_k$  as the classified  $k$ th reward amount or a classified  $k$ th reward inter-occurrence time such that  $\rho_k \in A$  for  $1 \leq k \leq n$  and  $\rho_k \in B$  for  $n+1 \leq k \leq 2n$ . Thus,  $\rho_k$ ,  $1 \leq k \leq 2n$  indicates a tier associated with the  $k$ th reward. Then, a set of events  $\cup \omega_{\rho_1, \dots, \rho_{2n}}$  is a collection of possible sequences of classified events for a wager alternative, where the events are mutually exclusive. Let  $P\{\omega_{\rho_1, \dots, \rho_{2n}}\} = p_{\rho_1 \dots \rho_{2n}}$ . And let  $Q = \{(\rho_1, \dots, \rho_{2n})\}$  be a collection of possible sequences of the reward tiers. Then, a probability measure for

a sequence of events can be expressed by a variable that satisfies

$$\sum_Q p_{\rho_1 \dots \rho_{2n}} = 1 \quad (5.1)$$

where  $p_{\rho_1 \dots \rho_{2n}} \geq 0$  for each sequence of  $\rho_k$ ,  $1 \leq k \leq 2n$ . For example, suppose that a reward process for a wager alternative is  $\{(X_k, \tau_k) : k = 1, 2\}$ , where  $X_k$  describes the  $k$ th reward amount while  $\tau_k$  expresses the inter-occurrence time between the  $k-1$ th and the  $k$ th rewards such that  $\tau_k = T_k - T_{k-1}$  and  $T_0 = 0$ . Let  $\{-2 \leq X_1 < -1\}$ ,  $\{1 \leq X_2 < 2\}$ ,  $\{1 \leq \tau_1 < 2\}$ ,  $\{\tau_2 < 1\}$  be tiers  $\rho_1 = -1$ ,  $\rho_2 = 2$ ,  $\rho_3 = 2$ ,  $\rho_4 = 1$  respectively. Then a probability measure for a sequence of these events  $P\{-2 \leq X_1 < -1, 1 \leq X_2 < 2, 1 \leq \tau_1 < 2, \tau_2 < 1\}$  can be expressed by  $p_{-1221}$ . Equation (5.1) is a necessary constraint corresponding to (4.3).

We now reformulate constraints  $\mathcal{A}'$  of formulations (4.1) and (4.2) using the discretized scheme. We first determine tiers  $\rho_k$ ,  $1 \leq k \leq 2n$ . For the  $k$ th reward amount  $a_k \in \mathbb{R}$ , we determine  $\rho_k = \lceil a_k \rceil = \min\{n \in \mathbb{Z} | a_k \leq n\}$ , and for the  $k$ th inter-occurrence time  $t_k \in \mathbb{R}_+$ ,  $\rho_k = \lceil t_k \rceil = \min\{n \in \mathbb{N} | t_k \leq n\}$ . Then the following event set can be converted to the union of a sequence of events restricted to  $\rho_k \leq \lceil v_k \rceil$ , such as

$$\{X_k \leq v_k\} \Rightarrow \bigcup_{Q | \rho_k \leq \lceil v_k \rceil} \omega_{\rho_1, \dots, \rho_{2n}}.$$

Probability assessments for this set provide the conversion form of the distribution function. With distributional bounds  $c_1$  and  $c_2$ , constraint (4.4) can be rewritten in discretized form

$$c_1 \leq P\{X_k \leq v_k\} \leq c_2 \quad \Rightarrow \quad c_1 \leq \sum_{Q | \rho_k \leq \lceil v_k \rceil} p_{\rho_1 \dots \rho_{2n}} \leq c_2 \quad (5.2)$$

for constants  $c_1, c_2 \in [0, 1]$ . In the same way, other constraints (4.5)–(4.7) can be expressed as discretized constraints by partitioning the events and assigning variables



for the probability measures on these events.

Next, we present a discretized form of the objective functions in (4.1) and (4.2). Let  $\tau_k$ ,  $k = 1, \dots, n$  be the inter-occurrence time as noted before. Let  $g(X_k, \tau_k)$  be a discount function for the  $k$ th reward such that  $g(X_k, \tau_k) = X_k e^{-\delta \sum_{j=1}^k \tau_j}$  with rate  $\delta$ . Using tiers for the amounts and the occurrence times of the rewards, the NPV of the total rewards with this discount function can be expressed as  $Z'$  such that

$$Z' = \sum_{k=1}^n \rho_k e^{-\delta \sum_{j=n+1}^{n+k} \rho_j}.$$

Using the discretization scheme, the expected utility can be converted to the following discrete form with variables  $p_{\rho_1 \dots \rho_{2n}}$ :

$$\sum_Q \left( u \circ Z' \right) p_{\rho_1 \dots \rho_{2n}}. \quad (5.3)$$

The mathematical programming formulations that correspond to the constrained calculus of variations (4.1) and (4.2) are constructed with objective function (5.3) and constraints, such as (5.1) and (5.2). These formulations for wager alternative  $\alpha$  can be expressed in a standard form as follows

$$\min h(\mathbf{p}_\alpha) \quad (5.4)$$

$$\text{s.t.} \quad \kappa_\alpha(\mathbf{p}_\alpha) \leq 0$$

for the lower bound of the expected utilities that corresponds to (4.1) and

$$\max h(\mathbf{p}_\alpha) \quad (5.5)$$

$$\text{s.t.} \quad \kappa_\alpha(\mathbf{p}_\alpha) \leq 0$$

for the upper bound of the expected utilities that corresponds to (4.2). In this formulation,  $h(\mathbf{p}_\alpha)$  represents the objective function (5.3), which is the expected utility of

the NPV of the total rewards associated with wager alternative  $\alpha$ . And  $\kappa_\alpha(\mathbf{p}_\alpha) \leq 0$  is all of the constraints, including both linear and nonlinear constraints, for alternative  $\alpha$ . The above formulations use variables  $\mathbf{p}_\alpha$  for the probability measures on sequences of categorized events and generate piecewise constant risks from their solutions. The resulting risks approximate the distribution functions that contribute to the maximum and the minimum expected utilities of the original formulations (4.1) and (4.2). Thus, interval  $I_\alpha = [l_\alpha, u_\alpha]$ , within which the expected utilities associated with wager alternative  $\alpha$  must lie, can be identified approximately by solving formulations (5.4) and (5.5).

Constraint (5.1) shows that the feasible solutions are on a hyperplane. Each constraint, except (5.1) and independence, introduces a halfspace on a given hyperplane. So the feasible region restricted by these constraints appears as a convex polyhedral set. The independence constraint may introduce non-convexity in this set. The addition of more constraints makes the feasible region smaller. If there is only one point in a feasible region, we have a unique probability law that is defined by that point. This implies that an unique distribution function is enough to characterize the uncertainty of alternative  $\alpha$  and that it has a single expected utility  $l_\alpha = u_\alpha$ . If the feasible region satisfying all of the constraints is empty, there is no distribution function that characterizes the uncertainty of the alternative. In this case, some constraints must be revised for building a consistent set of constraints.

Not only the independence constraint, but also a complicated utility function, provide nonlinearity in formulations (5.3) and (5.4). Thus, the formulations from discretization are often nonconvex and nonlinear. In addition, the size of these programming formulations is generally quite large since they have  $(N_1 \cdot N_2)^n$  variables if  $N_1$  and  $N_2$  are the number of tiers of the reward amounts and inter-occurrence

times.<sup>1</sup> For example, suppose that there is a wager alternative whose rewards form the process  $\{X_k, \tau_k\}$ ,  $k = 1, \dots, 5$ . If we classify the reward amounts and the reward inter-occurrence times into 10 tiers respectively, the mathematical programming problems that correspond to (5.4) and (5.5) must contain  $(10 \cdot 10)^5 = 10$  billion variables. Therefore, application of the formulations introduced in this chapter often results in non-convex, nonlinear problems along with a sizeable number of variables. Heuristic solution methods for solving various classes of non-convex and nonlinear programming formulations are abundant. However, our formulation has some especially convenient features: 1) the feasible region can be expressed by a set of restricted probability measures, and 2) the constraints,  $\kappa_\alpha$ , have a rather simple structure in as much as they are typically smooth. With these features, it is possible to develop highly stylized algorithms with potential for improvement in computational efficiency beyond the general methods. In the next section, we introduce such methods that are applicable for the discretized programming problems (5.4) and (5.5).

## B. A Solution Method for Optimization Problems in Wager

Discretized programming is formulated with  $(N_1 \cdot N_2)^n$  variables  $p_{\rho_1 \dots \rho_{2n}}$  whose feasible values form a distribution function. The expected utilities corresponding to these variable values are then computed in an objective function of the formulation. These expected utilities approximate the original expected utility bounds modeled in the constrained variational calculus (4.1) and (4.2). The method for estimating the expected utility bounds described here consists of two separate inner algorithms: one is used to update  $p_{\rho_1 \dots \rho_{2n}}$  to satisfy the constraints, and the other is used to evalu-

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<sup>1</sup>Cardinality of sets  $A$  and  $B$ .

ate the expected utilities with the given  $p_{\rho_1 \dots \rho_{2n}}$ . Both algorithms utilize the idea of *Monte Carlo integration* that was originally introduced to approximate the value of high dimensional integrals using samples [55]. The former searches the better feasible values with a higher probability while the latter approximates the expected utilities with samples from an irreducible Markov Chain. We now describe these algorithms.

### 1. An algorithm for feasible solution exploration

The algorithm for generating feasible values capitalizes on the characteristics of a Markov chain for sampling. If a Markov chain is built appropriately, samples with a better expected utility might be available with higher probability. Suppose that we have a Markov chain whose transition probability  $\pi^\theta$  is defined by

$$\pi^\theta(\{p_{\rho_1 \dots \rho_{2n}}\}) = \frac{\exp\{\theta \cdot h(\mathbf{p}_\alpha)\}}{C_\theta} \quad (5.6)$$

where constant  $\theta > 0$ , and  $C_\theta$  is the sum of the numerators for all  $\{p_{\rho_1 \dots \rho_{2n}}\}$  satisfying constraints  $\kappa_\alpha$ . In this transition probability,  $C_\theta$  plays the role of an unknown normalizing constant. Discretized expected utility (5.5) is used in this transition probability. Let  $P_t$  be a set of values for variables  $p_{\rho_1 \dots \rho_{2n}}$  at the  $t$ th iteration. This set represents a distribution function in discretized form that satisfies the constraints at iteration  $t$ . Suppose that  $\mathcal{B}_\alpha$  is a set of distribution functions that satisfy the constraints for alternative  $\alpha$  with variables  $p_{\rho_1 \dots \rho_{2n}}$  such that  $\mathcal{B}_\alpha = \{\mathbf{p}_\alpha : \kappa_\alpha(\mathbf{p}_\alpha) \leq 0\}$ , corresponding to  $\mathcal{A}'_\alpha$  in (4.1) and (4.2). We develop a *Metropolis algorithm* [31] that samples the next feasible distribution  $P_{t+1}$  based on the Markov chain having the above transition probability. Its pseudo code is now described below.

### ***A MCMC optimization algorithm for feasible solution explo-***

***ration***

Set  $t = 0$

Initialize  $P_t$

Evaluate the objective function with  $P_t$

Repeat unless the termination condition is satisfied {

- (1) Choose two variables  $J_+$  and  $J_-$  from  $P_t$ .
- (2) Sample  $Y$  from a proposal distribution  $\phi(\cdot|J_+)$ .
- (3) Formulate a set of variables  $P_M$  by changing  $J_+$ ,  $J_-$  in  $P_t$  into  $Y$  and  $J_- + (J_+ - Y)$ , respectively.
- (4) If  $P_M \notin \mathcal{B}_\alpha$ , then  $P_{t+1} = P_t$  and go to (6).
- (5) If  $P_M \in \mathcal{B}_\alpha$ , generate a  $U(0,1)$  and let the acceptance probability  $\gamma$  be
 
$$\gamma = \min \left\{ 1, \frac{\pi^\theta(P_M)}{\pi^\theta(P_t)} \right\}$$
 If  $U \leq \gamma$ , then  $P_{t+1} = P_M$  and evaluate the objective function; otherwise,  $P_{t+1} = P_t$ .
- (6) Increment  $t$ .

}

The state space for the Markov chain in the above algorithm is bounded in  $[0, 1]$ . Proposal distribution  $\phi$  can be any distribution but  $Y$ ; a sample from  $\phi$ , must be in  $[0,1]$ . For this distribution, we use a normal distribution with mean  $J_+$  and an adjustment that is necessary for  $Y$  to be in  $[0, 1]$ . Step (3) generates the candidate solution  $P_M$  that satisfies probability condition (5.1). The normalized constant  $C_\theta$  is set off in the computation of the acceptance probability  $\gamma$ . Let  $(\eta_1 \cdots \eta_{2n})$  be

the index sequence of  $J_+$  such that  $J_+ = p_{\eta_1 \dots \eta_{2n}}$ . And let  $(\zeta_1 \dots \zeta_{2n})$  be the index sequence of  $J_-$  such that  $J_- = p_{\zeta_1 \dots \zeta_{2n}}$ . Then the acceptance probability  $\gamma$  is simplified when rearranged with the objective function  $h(\mathbf{p}_\alpha)$ . Let  $Z'$  be the NPV of total rewards associated with a wager alternative with a discount function  $g(X_k, \tau_k)$  such that  $Z' = \sum_{k=1}^n \rho_k e^{-\delta \sum_{j=n+1}^{n+k} \rho_j}$ . The acceptance probability with  $Z'$  can be rewritten by

$$\begin{aligned} \gamma &= \min \left\{ 1, \frac{\pi^\theta(P_M)}{\pi^\theta(P_t)} \right\} \\ &= \min \left\{ 1, \exp \left\{ -\theta \left( u \circ \sum_{k=1}^n \eta_k e^{-\delta \sum_{j=n+1}^{n+k} \eta_j} \right) (Y - J_+) \right. \right. \\ &\quad \left. \left. + \left( u \circ \sum_{k=1}^n \zeta_k e^{-\delta \sum_{j=n+1}^{n+k} \zeta_j} \right) (J_+ - Y) \right\} \right\}. \end{aligned}$$

To adjust the convergence speed to the equilibrium distribution  $\pi^\theta$ , it is possible to set up  $\theta(t)$  instead of the fixed  $\theta$ . In this case, the chain becomes a *time inhomogeneous* Markov Chain. If the candidate  $P_M$  is accepted, the updated distribution function generates an improving expected utility value that moves along the variable coordinate axes. The algorithm goes with the evaluation of the objective function whenever the solutions are updated. However, evaluating the expected utility may require a staggering amount of computational time, because the number of variables we need to consider is  $(N_1 \cdot N_2)^n$ , where  $N_1$  and  $N_2$  are the number of tiers for events associated with the reward amounts and the reward inter-occurrence times. Hence, even a small  $n$  can easily result in a large scale calculation for this evaluation. Consequently, we recognize the necessity for developing an algorithm that approximates the objective function in a shorter time compared to the time for evaluating the objective function for all of the variables. An algorithm for this purpose is described in the next section.

## 2. An algorithm for objective function evaluation

We develop an *Importance sampling* algorithm, one of the Monte Carlo integrations, that is viable for the approximation of an expected utility with a given  $P_t$ , a distribution function in the discretized form. Suppose there exists a distribution  $Q$  such that  $G(\underline{y})$  has a density  $g(\underline{y})$  with respect to  $Q$ . Then, by the Radon-Nikodym theorem, the corresponding derivative is  $dG/dQ = g$ . Suppose that a distribution function of the NPV  $Z_\alpha$  of the total rewards associated with wager alternative  $\alpha$  is  $G(\underline{y})$ ,  $\underline{y} = (y_1, \dots, y_{2n})$ . Then, the objective function of the mathematical formulations (4.1) and (4.2) can be rewritten as

$$E[u \circ Z_\alpha] = \int_{\mathbb{R}^n \times \mathbb{R}_+^n} (u \circ z_\alpha)(\underline{y}) g(\underline{y}) dQ(\underline{y})$$

with respect to distribution function  $Q$ . Suppose we have  $P_t$ , a distribution at iteration  $t$  for  $g(\underline{y})$ . Then, we can approximate the above expected utility by the importance sampling formulation

$$E[u \circ Z_\alpha] \approx \frac{1}{m} \sum_{i=1}^m \left\{ u \circ \sum_{k=1}^n s_k^i e^{-\delta \sum_{j=n+1}^{n+k} s_j^i} \right\} p_{S^i} \quad (5.7)$$

with  $m$  samples  $S^i = (s_1^i \cdots s_{2n}^i)$ ,  $i = 1, \dots, m$  from the distribution  $Q$ . The distribution  $Q$  can be any probability function regardless of  $P_t$ . This sampling method avoids complete expectation calculation with all components of  $P_t$  by using the samples in (5.7). This lightens the computational burden of evaluating the objective function. Suppose that formulations (5.4) and (5.5) include  $n$  variables. If we evaluate the objective function through the algorithm (5.7) with  $m$  samples, the computation time can be reduced by  $m/n$  of the direct evaluation. The elements of  $P_t$ , updated through the MCMC optimization algorithm, must be included in the samples. If this condition

fails, it may be possible to use the same evaluation values as those evaluated by the old distributions. This condition can be met easily by using the same samples for each  $P_t$ ,  $t = 1, \dots$  and only alternating two for the updated elements. The availability of identical samples over iterations of the evaluation algorithm further strengthens the computational efficiency when compared to other sampling methods. The complexity of this double MCMC algorithm is  $3 \log n + \mathcal{O}(n^k) + \mathcal{O}(3n \log n + n^{k+1} + n^{k-1})$  when the discretized mathematical problems have a complexity  $\mathcal{O}(n^k)$ . The complexity of the problems (5.4) and (5.5) is determined by the number of variables and the discount function. Thus, the described algorithms have only one more degree of complexity than the objective function's.

A solution method that includes a Importance sampling for the approximation of the expected utility and a MCMC optimization algorithm for searching the feasible solutions is introduced for optimization problems (5.4) and (5.5). This method is designed to find distribution functions that minimize or maximize the expected utilities in the discretized models. If we consider proximity to the optimal solution<sup>2</sup> as a termination condition for the MCMC optimization algorithm, this method satisfies this condition in polynomial time. If an exponential discount function is applied to the NPVs of the total rewards with  $n$  variables, the complexity of the given method is  $\mathcal{O}(n^4)$  while the discretized problems have a complexity  $\mathcal{O}(n^3)$ . The algorithm has only one more degree of complexity than the objective function's, e.g., if a complexity of the objective function is  $\mathcal{O}(n^k)$  for  $k \in \mathbb{N}$ , a complexity of the algorithm becomes  $\mathcal{O}(n^{k+1})$ . This shows the computational efficiency of our algorithms. In the next section, the identification of the minimum and maximum expected utilities of formu-

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<sup>2</sup> Given  $\epsilon > 0$ , if difference between the global optimum  $P^*$  and the solution  $P_t$  is less than  $\epsilon$ , that is,  $|P^* - P_t| < \epsilon$ , the algorithm progress is stopped.



lations (4.1) and (4.2), using the discretized formulations, are discussed.

### C. Identification of Expected Utility Bounds

If solutions of the discretized mathematical programming problems are obtained, these provide an approximation of the interval bounds associated with the original constrained calculus of variations (4.1) and (4.2). The reason is that the discretized optimization problem is constructed by classifying the events related to the reward amounts and the inter-occurrence times. However, solutions that either exceed the minimum expected utility or do not reach the maximum expected utility are inept at describing the interval bounds for the purpose of wager alternative separation. With such solutions, a wager alternative may be falsely considered separable from others even though there exists an intersection with intervals whose bounds are congruent with the original variational calculus (4.1) and (4.2). Thus, the approximated interval bounds must cover an interval that is comprised of the minimum and maximum expected utilities. This can be accomplished by an adjustment in the discretization scheme.

Let  $I_\alpha = [l_\alpha, u_\alpha]$  be the interval for wager alternative  $\alpha$ . Then the solutions that approximate the interval bounds through the discretized formulation might be positioned within boundaries, say  $[a, b]$  for  $l_\alpha$  and  $[c, d]$  for  $u_\alpha$ , as shown in Figure 3. The interval for alternative  $\alpha$  can then be expressed by identifying  $a$  and  $d$ . Suppose that we have obtained the minimum and maximum expected utilities,  $u_{min}$  and  $u_{max}$ , from the discretized formulation. The obtained  $u_{min}$  and  $u_{max}$  can be considered as upper bounds of  $l_\alpha$  and  $u_\alpha$ , say  $b$  and  $d$ , because the discretization utilizes ceiling functions. Suppose that we have a constraint on a reward probability,

such as  $c_1 \leq P\{X \leq 4.5\} \leq c_2$  with constants  $c_1, c_2 \in [0.1]$ . By classifying the event as described in section A, we can observe

$$\sum_{\rho=1}^4 p_{\rho} \leq P\{X \leq 4.5\} \leq \sum_{\rho=1}^5 p_{\rho}$$

where  $p_{\rho}$  represents the probability that the reward belongs to tier  $\rho$ . This can be generalized by

$$\sum_{Q_{|\rho_k \leq \lfloor v_k \rfloor}} p_{\rho_1 \dots \rho_{2n}} \leq P\{X_k \leq v_k\} \leq \sum_{Q_{|\rho_k \leq \lceil v_k \rceil}} p_{\rho_1 \dots \rho_{2n}}$$

where  $\lfloor v_k \rfloor = \max\{n \in \mathbb{Z} | n \leq v_k\}$  for  $1 \leq k \leq n$ , and  $\lceil v_k \rceil = \max\{n \in \mathbb{N}_+ | n \leq v_k\}$  for  $n+1 \leq k \leq 2n$ . The discretized formulation with the right-hand side of the above inequality has been explained in section A. If we set up constraints with the left-hand side, the bounded marginal distribution (4.4) can be rewritten in a discretized form

$$c_1 \leq P\{X_k \leq v_k\} \leq c_2 \quad \Rightarrow \quad c_1 \leq \sum_{Q_{|\rho_k \leq \lfloor v_k \rfloor}} p_{\rho_1 \dots \rho_{2n}} \leq c_2 \quad (5.8)$$

The other constraints, (4.5)–(4.7), can be converted to discretized forms in the same way. Then, the solutions from the objective function (5.5) and these constraints provide the lower boundaries of the set of expected utilities,  $a$  and  $c$  for the interval bounds  $l_{\alpha}$  and  $u_{\alpha}$  such that  $l_{\alpha} \in [a, b]$ ,  $u_{\alpha} \in [c, d]$ . In order to identify the boundaries  $a, d$  for the wager alternative satisfying  $[l, u] \subset [a, d]$ , we need only to implement a maximization programming problem (5.5) with constraints applying the ceiling function for upper boundary  $d$  and then implement a minimization programming problem (5.4) with constraints applying the floor function for lower boundary  $a$ . The solution method presented earlier approximates the solutions of the original optimization models,  $l$  and  $u$ , which then consist of the interval  $[a, d]$ . Additionally, it is obvious that more tiers in a classification make the approximated interval  $[a, d]$  closer to the

interval  $[l, u]$  but expand the computational burden by the increased number of variables.



Fig. 3. Boundaries  $a$ ,  $b$ ,  $c$  and  $d$  for interval  $I_\alpha = [l_\alpha, u_\alpha]$  such that  $l_\alpha \in [a, b]$  and  $u_\alpha \in [c, d]$ .

#### D. Wagering with MCMC Optimization Algorithm

Our focus in this research is on how to find the best alternative based on risk preference among marked point process wager alternatives. As noted, wager alternatives can be analytically partitioned by representing the set of expected utilities associated with alternatives on an interval graph and adopting the principles described in chapter IV. Practical application of the principles involves the identification of the interval  $[a, d]$  using the presented solution method. However, if a parallel processing system can be available, we can achieve our purpose by avoiding redundant calculations.

The MCMC optimization algorithm updates a solution moving inside the feasible region toward the optimum that maps into boundary  $a$  or  $d$ . That is, the expected utility mapped from the solution approaches these boundaries from the internal points on the interval. On the other hand, the estimate for the expected utility approaches the boundaries from outside the interval  $[a, d]$  while proceeding with the Importance sampling algorithm for objective function evaluation. Thus, when the solutions found

by MCMC optimization go to *stable*<sup>3</sup> at a certain moment, we can observe the separation of one interval from the others while processing the evaluation of the expected utility. In such a case, further work for a lesser preferred wager alternative is unnecessary. Because this research focuses on finding the best alternative among the wager alternatives, we also stop the process for wager alternatives that are separated from a maximal clique that includes the alternative with the greatest lower bound. This will lessen the computational burden for a wager whose alternative rewards are complicated marked point processes where we are unable to characterize the uncertainty as an unique risk.

For the nonseparable alternatives, we can classify the events into more tiers and apply a procedure previously illustrated for additional separation. This method has a chance of success because the revised classification makes interval  $[a, d]$  closer to interval  $[l, u]$ . This is useful when the value of alternative separation is greater than the opportunity costs of retrying discretization and problem solving. In the next section, we present an example of the application of a solution algorithm for these optimization problems and an interpretation of the outputs related to the separation of risks associated with the wager alternatives.

#### E. An Example: Separation of Wager Alternatives Using MCMC Optimization

We consider simple wager alternatives whose rewards appear over discrete time. Suppose that the rewards from wager alternative  $\alpha$  can be represented as two random variables  $X_1$  and  $X_2$  for the reward amounts at the end of year 1 and year 2. Define these random variables on the probability space  $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ , where  $\mathcal{F}_\alpha$  is the natural

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<sup>3</sup> The circumstance of the difference  $|h(P_t) - h(P_{t+1})| < \delta$  over quite a number of iterations with a fixed  $\delta > 0$

filtration with the  $\sigma$ -algebra of these variables. Then, the expected utility associated with the joint distribution function for  $X_1$  and  $X_2$  is defined by

$$E[u \circ Z_\alpha] = \int_{\mathbb{R}^2} (u \circ z_\alpha)(\underline{y}) dG_\alpha(\underline{y})$$

where  $G_\alpha(\underline{y}) = P\{X_1 \leq y_1, X_2 \leq y_2\}$ . And  $Z_\alpha$  is the NPV with  $X_1$  and  $X_2$ . Suppose that we have some information about the probabilities of these rewards that will restrict the set of distribution functions,  $\mathcal{A}'_\alpha$  such that  $G_\alpha \in \mathcal{A}'_\alpha$ . Then, a set of feasible risks, subject to our level of understanding, for alternative  $\alpha$  can be constructed with the constraints shown in Appendix A. With the objective function above, it is possible to set up constrained variational calculus for identifying the bounds on the set of the expected utilities into which the risk of alternative  $\alpha$  maps.

We now setup a standard mathematical programming problem using a discretization scheme. With the classification of 10 tiers from 1 to 10 for each reward, the original variational calculus is converted into an optimization problem with variables  $p_{\rho_1\rho_2}$ ,  $\rho_j \in \{1, \dots, 10\}$ ,  $j = 1, 2$ . Suppose that we have a linear utility function  $u(w) = w$  and an exponential discount function with  $\delta = 0.05$ . Then the objective function in a discretized form is

$$h(\mathbf{p}_\alpha) = \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^{10} (\rho_1 e^{-\delta} + \rho_2 e^{-\delta^2}) p_{\rho_1\rho_2} \quad (5.9)$$

with constraints shown in Appendices B and C. The constraints in Appendix B adopt the ceiling function for classification, and they are used for finding the upper boundary of the expected utilities while the constraints in Appendix C adopt the floor function for classification, and are used to determine the lower boundary. The MCMC optimization algorithm, using MATLAB, is applied to the above problems to find boundaries  $a$  and  $d$  in Figure 3. In each iteration, we select  $J_+$  and  $J_-$  uniformly from among the 100 variables, and sample  $Y$  from a normal distribution with mean

$J_+$  and variance 0.02. The best solutions are chosen through 5,000 iterations of this model. Since the formulation in discretized form for this example is linear, the solutions using the MCMC algorithm are verified by a linear programming code, AMPL, in CPLEX. The solutions and their corresponding cumulative distributions are illustrated in Appendix G. For additional alternative  $\beta$  with constraints in Appendix D, we develop discretized formulations and apply the MCMC algorithm as shown for alternative  $\alpha$ . The discretized formulations for alternative  $\beta$  are given in Appendices E and F. Their solutions are described in Appendix G.

The obtained solutions are probability distributions that contribute to the boundaries that cover a set of expected utilities into which the risk of each alternative maps. Figure 4(a) illustrates two marginal distributions of  $X_1$  with classified tiers  $\rho_1$ . The marginal distributions that capture the dynamics of the marked point process from alternative  $\alpha$  must fall between these two distributions. The boundaries corresponding to these two distributions are shown in Figure 4(b). For alternative  $\beta$ , figure 5(a) and (b) show the two marginal distributions and their expected utilities as boundaries. The concluding interval for wager alternative  $\alpha$  is  $[a_\alpha, d_\alpha] = [8.4529, 11.0445]$ , while it is  $[a_\beta, d_\beta] = [7.3036, 10.6313]$  for alternative  $\beta$ . The risk of the alternative must be included in the sets of distribution functions whose expected utilities are bounded in each interval. For wager alternatives  $\alpha$  and  $\beta$ , we cannot separate the risk of alternative  $\alpha$  from alternative  $\beta$  based on risk preference since their corresponding intervals are partially overlapped.

Suppose that we have five more alternatives  $A, B, C, D$ , and  $E$  as described in chapter III. The expected utilities of these alternatives are  $E_A = 5$ ,  $E_B = 8.333$ ,  $E_C = 9.7376$ ,  $E_D = -1.44$ , and  $E_E = 5.04$ , respectively. Among these seven alternatives, alternative  $C$  has the greatest lower bound 9.7376. The maximal clique that includes

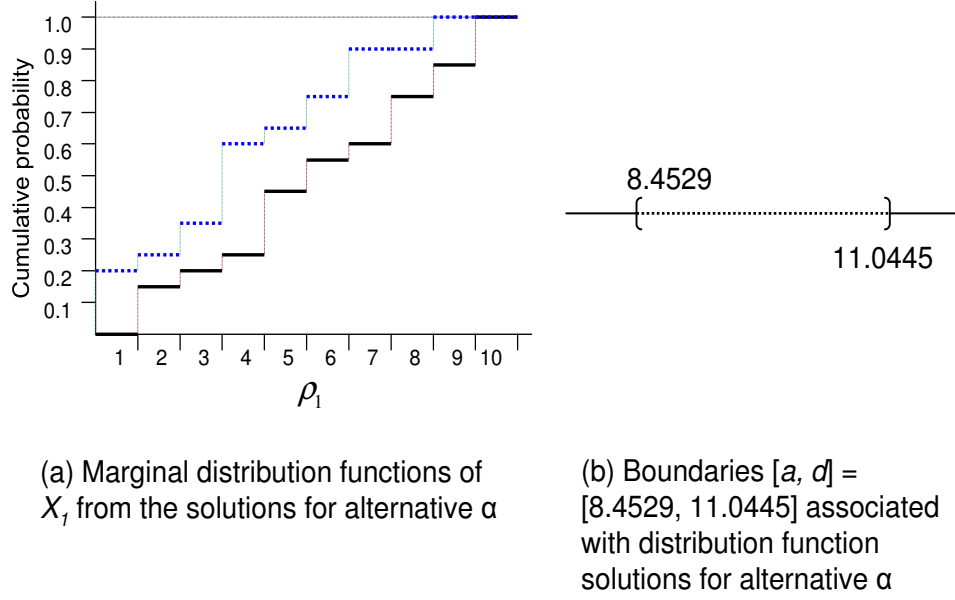


Fig. 4. Distributions and the corresponding boundaries on the expected utility for wager alternative  $\alpha$ .

alternative  $C$  consists of alternatives  $\alpha$ ,  $\beta$ , and  $C$ . By the corollary in chapter IV, alternative  $C$  is preferable to alternatives  $A, B, D, E, F$ , while alternatives  $\alpha$  and  $\beta$  are indistinguishable from alternative  $C$ .

This chapter has presented a numerical approach for separating preferences among marked point process wager alternatives. We developed a discretization scheme for constructing standard mathematical programming problems that is numerically applicable for the identification of a set of expected utilities. These formulations are often nonlinear, nonconvex and large-scale. We introduced a solution method stylized for such formulations that may appear in a wager. The method, which consists of two parts, a Metropolis algorithm for searching feasible solutions and a Importance sampling method for evaluating objective functions, is a polynomial time algorithm with only one more degree of complexity than the complexity of the problems, e.g., for problem complexity  $\mathcal{O}(n^k)$ , the algorithm complexity is  $\mathcal{O}(n^{k+1})$ . By employing

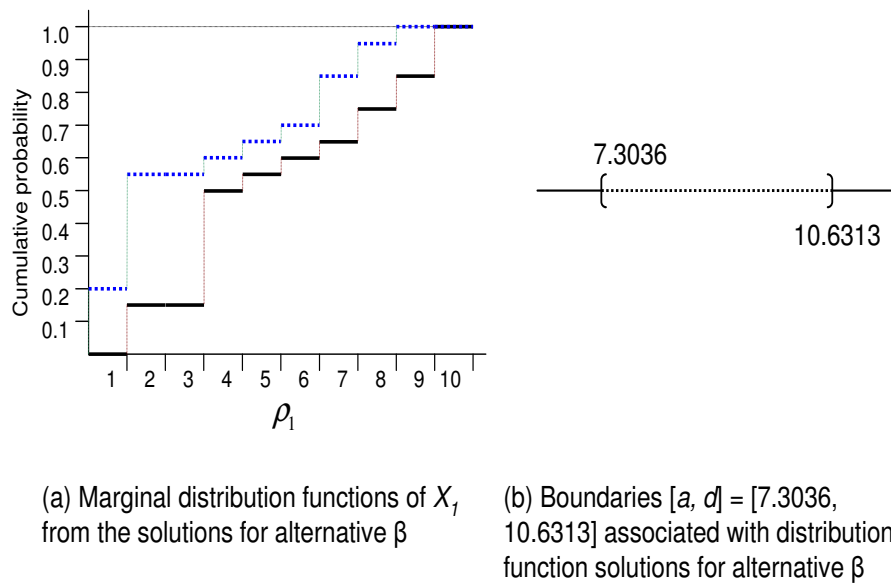


Fig. 5. Marginal distributions and the corresponding boundaries on the expected utility for wager alternative  $\beta$ .

this method, the interval corresponding to a set of expected utilities can be identified and we can determine which alternative is separable from the others and has the most preferred risk. We have also discussed the insight that when several processors are simultaneously used, wagering with the MCMC optimization algorithm may cause a reduction in computational effort since the objective of our research is not to identify the set of expected utilities for all alternatives but to find the alternative with the most favorable risk among the wager alternatives. An example is introduced to show an application of the numerical approach for a wager whose alternative rewards may be a complicated marked point process.



## CHAPTER VI

### CONCLUSION

In this research, we have focused on wagers, which are a one-time decision of alternatives that have a sequence of uncertain outcomes. By presenting mathematical formulations and a procedure, we have built a fundamental framework for addressing which alternative is the most favorable by risk preference among wager alternative outcomes which are modeled in a marked point process. Additionally, we presented a numerical approach available for an application of this framework in a practical situation.

In wager problems where decision is to be made based on risk preference, well-known discrete distributions or normal distributions are assumed to capture the dynamics of random rewards. No study has been conducted on wager problems where only restrictive data on random rewards are available and where the random rewards appear over a long period of time. In addition, a general analytical framework for the selection of preferences among wager alternatives has yet to be introduced.

We have explored the risk associated with a marked point process wager alternative and the implication of the EUT in a wager in chapter III. For wager alternatives with a large number of random rewards, it is difficult to evaluate the value of the alternatives even in terms of expected utilities. But in cases where the alternative rewards can be represented as well-known stochastic processes, we can obtain the risks and their corresponding expected utilities. We employed the SLLN to verify the existence of such expected utilities and the CLT to estimate them. Finding the alternative with the most favorable risk from among such wager alternatives can be accomplished by comparing their expected utilities. For wager alternatives only restrictive data on rewards are available, an unique risk is not enough to characterize

the uncertainty. We represent non-unique risk as a set of distribution functions and introduce an interval for this set. A mathematical formulation, that is a calculus of variation form, for identifying the interval bounds was devised in chapter IV. This formulation utilizes our prior understanding of the probability measures on the rewards in constraints. We developed a procedure that includes a lemma and a corollary to determine the most favorable alternative among wager alternatives under a non-unique risk characterization. This reveals a situation no decision is made. The procedure with a mathematical formulation provides us a fundamental framework for selection of preferences in a wager. As noted in chapter V, solving the variational calculus problems must lean for partial differential equations only some of which are known to be solvable. We have, therefore, constructed a standard mathematical programming problem that corresponds to a variational calculus. The reformulated formulation is generally nonlinear, nonconvex, and large-scale. Thus we have developed a solution method that is stylized based on features of the problems for improvement in computational efficiency. This solution method has a polynomial time complexity with only one more degree of complexity than does the problem. Applying the numerical approach, that consists of the reformulation and the solution method, for a wager, the most favorable alternative can be obtained among wager alternatives that have a sequence of random rewards. Finally, we have discussed that our numerical approach may cause a reduction in computational effort when wagering.

For further research, we introduce two open issues: 1) sensitivity analysis regarding constrained variational calculus formulations for alternatives, 2) stationary process wager alternatives that may offer a simple formulation of the expected utility. Following the discussion of further research, we conclude this dissertation by discussing viable application areas for our findings.

## A. Future Research

### 1. Sensitivity analysis

Sensitivity analysis related to the separation of risk preferences among wager alternatives includes the analysis of constraint relaxation which supports the bounds of the alternative's expected utilities. Such analysis generally offers insight into how the events that constitute the boundaries of a feasible region can be identified. However, optimization problems, even in a discretized form, often become nonlinear, nonconvex, and NP-hard. Hence, the sensitivity analysis required for a wager turns out to be very complicated in such cases. Nonetheless, this analysis is important because it can explain the relationship between the cost required for information acquisition about reward events and the possibility of additional separation among the most preferred alternatives. By investing some money, we may be able to rule out an alternative that was indivisible in a maximal clique with the greatest lower bound of the interval. In addition, we can evaluate the worth of information about alternative rewards based on the importance of a supplementary separation of preferences among the alternatives. Thus, sensitivity analysis may provide a guideline for deciding an acceptable level of investment for obtaining more information about the rewards accrued by the most preferred wager alternatives.

For sensitivity analysis in a wager, we want to develop a standard procedure tailored to an overall wagering process with connections to the value of the information on reward events. This procedure must be able to guarantee alternative separation not only in discretized formulation but also in the original variational calculus.

## 2. On stationary marked point process rewards

For stationary ergodic process reward alternatives, the variances must be known before applying asymptotic normality for the NPV of the rewards. Markov chain and martingale rewards experience the same situation. The variances in these cases are conventionally estimated by sampling techniques such as *standard time series* [24, 25, 26, 27] or *window estimators* [30, 28]. Nonetheless, it is well-known that finding proper variances is challenging and time-consuming. For stationary process wager alternatives, we can avoid this cumbersome task by identifying the relationship between the NPV of the stationary process rewards and applying *Campbell's formula* [56, 57]. With Campbell's formula, we obtain the expected utility for a stationary stochastic process wager alternative in a different direction than when employing CLT. In addition, it is viable for a small number of stationary process reward alternatives that are not suited to CLT adoption.

Suppose we have a stationary marked point process  $\{(Y_n, S_n), n \in \mathbb{N}\}$  with non-negative random variables  $Y_n$  for reward amounts, and  $S_n$  for reward occurrence time. Here, each random variable is on probability space  $(\Omega, \mathcal{F}, P)$ , and there is no reward occurrence at time 0. Let  $\{(X_n, T_n), n \in \mathbb{N}\}$  be another stationary marked point process with  $P\{T_0 = 0\} = 1$ . Suppose that the process  $N^0$  is  $\{(X_n, T_n), n \in \mathbb{N}\}$ , the *Palm* version of  $\{(Y_n, S_n), n \in \mathbb{N}\}$ , which implies “ $\{(Y_n, S_n), n \in \mathbb{N}\}$  on the condition that it has a point at 0”. With a function  $g : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , by *Campbell's formula*, we have

$$E\left[\sum_{n=1}^{\infty} g(Y_n, S_n)\right] = \lambda \int_{\mathbb{R}_+} \int_{\mathbb{R}} g(x, t) P\{x_0 \in dx\} dt \quad (6.1)$$

$$= \lambda E[g(N^0)] \quad (6.2)$$

where  $\lambda = E[T_1]^{-1}$ , and  $P\{x_0 \in dx\}$  is a *Palm probability* with the reward amount  $x_0$

at time 0. If we consider  $g$  as the product of the utility and the discount functions, we can construct an equation (6.1) with signed Palm measures. The right-hand side of the equation is departed into two terms: the negative and the positive parts of the rewards. Since the discount function vanishes as time passes, function  $g$  decreases to zero, and then there exists the expectation in (6.2). This indicates that an expected utility associated with stationary marked point process rewards is obtained when evaluating a Palm version of the original process in a much simpler form, and the method is suitable for all stationary processes even those without an ergodicity property.

We can find the expected utility via formulation (6.2) by addressing the rigorous relationships between stationary marked point process rewards of wager alternatives and their Palm versions and the development of a practical form for the calculation of expected utility. Related further research must include a proper solution method for expected utility.

## B. Further Areas for Application

Technology assessment is one field where the results of this research should play a role in the future. The procedures described here can be used to model adoption plans with estimates of the expected revenues and costs for applying new technology to a system or product. Further, the estimations can be diversified to accommodate technology adoption plans. However, revenues and costs can only be obtained when decision makers understand the impact of adoption plans. Thus, if we represent the likely revenues and costs of each plan as a stochastic process, it will be possible to select the most favorable technology adoption plan using the frameworks and the

numerical approach in this dissertation. The value of the selected plans (e.g., the NPV of the plans selected for technology adoption) can then be considered as a practical assessment of the new technology. In addition, design flexibility and facility location problems can be handled by evaluating options that have a sequence of uncertain events with this research result.

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## APPENDIX A

CONSTRAINTS FOR WAGER ALTERNATIVE  $\alpha$ 

Wager alternative  $\alpha$  has random rewards at year 1 and year 2, say  $X_1$  and  $X_2$  on the probability space  $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ . Our level of understanding or data for probability measures on these rewards are arranged into three categories: bounded marginal distributions, bounded conditional distributions, and ordinal distributions. This information contributes to building a set of feasible risks  $\mathcal{A}'_\alpha$  as constraints. All constraints for wager alternative  $\alpha$  are described below. The numbers to the right of each line correspond to the constraint numbers.

## Bounded marginal distributions

$$0.180 \leq P\{X_1 \leq 2.0\} \leq 0.287 \quad (1)$$

$$0.480 \leq P\{X_1 \leq 4.5\} \leq 0.780 \quad (2)$$

$$0.670 \leq P\{X_1 \leq 7.8\} \leq 0.995 \quad (3)$$

$$0.668 \leq P\{X_1 > 3.0\} \leq 0.784 \quad (4)$$

$$0.335 \leq P\{X_1 > 5.5\} \leq 0.428 \quad (5)$$

$$0.090 \leq P\{X_1 > 7.3\} \leq 0.350 \quad (6)$$

$$0.080 \leq P\{X_1 > 8.5\} \leq 0.137 \quad (7)$$

$$0.165 \leq P\{X_2 \leq 2.0\} \leq 0.288 \quad (8)$$

$$0.205 \leq P\{X_2 \leq 2.5\} \leq 0.387 \quad (9)$$

$$0.410 \leq P\{X_2 \leq 4.5\} \leq 0.584 \quad (10)$$

$$0.603 \leq P\{X_2 \leq 7.0\} \leq 0.814 \quad (11)$$

$$0.654 \leq P\{X_2 > 2.3\} \leq 0.745 \quad (12)$$

$$0.550 \leq P\{X_2 > 3.7\} \leq 0.628 \quad (13)$$

## Bounded marginal distributions

$$0.350 \leq P\{X_2 > 5.8\} \leq 0.430 \quad (14)$$

$$0.150 \leq P\{X_2 > 7.5\} \leq 0.220 \quad (15)$$

$$0.080 \leq P\{X_2 \geq 9.5\} \leq 0.135 \quad (16)$$

$$0.010 \leq P\{X_1 \leq 2.0, X_2 \leq 2.0\} \leq 0.045 \quad (17)$$

$$0.135 \leq P\{X_1 \leq 3.5, X_2 \leq 3.9\} \leq 0.230 \quad (18)$$

$$0.238 \leq P\{X_1 \leq 4.7, X_2 \leq 5.3\} \leq 0.380 \quad (19)$$

$$0.460 \leq P\{X_1 \leq 6.2, X_2 \leq 7.6\} \leq 0.612 \quad (20)$$

$$0.825 \leq P\{X_1 \leq 8.3, X_2 \leq 9.1\} \leq 0.950 \quad (21)$$

$$0.120 \leq P\{X_1 \leq 3.0, X_2 > 3.4\} \leq 0.180 \quad (22)$$

$$0.150 \leq P\{X_1 \leq 6.5, X_2 \leq 2.3\} \leq 0.250 \quad (23)$$

$$0.130 \leq P\{X_1 \leq 8.0, X_2 > 8.0\} \leq 0.185 \quad (24)$$

$$0.480 \leq P\{X_1 \leq 8.1, X_2 \geq 3.2\} \leq 0.585 \quad (25)$$

$$0.045 \leq P\{X_1 \leq 5.5, X_2 \leq 1.0\} \leq 0.075 \quad (26)$$

$$0.032 \leq P\{X_1 > 7.5, X_2 \leq 2.0\} \leq 0.052 \quad (27)$$

$$0.035 \leq P\{3.7 \leq X_1 \leq 5.5, X_2 \leq 1.8\} \leq 0.048 \quad (28)$$

## Bounded conditional distributions

$$0.054 \leq P\{X_2 \leq 1.0 \mid X_1 \geq 2.0\} \leq 0.125 \quad (29)$$

$$0.450 \leq P\{X_2 \leq 6.0 \mid X_1 \geq 9.5\} \leq 0.850 \quad (30)$$

$$0.250 \leq P\{X_1 \leq 4.0 \mid X_2 \geq 9.0\} \leq 0.480 \quad (31)$$

$$0.320 \leq P\{X_1 \leq 4.0 \mid 4.5 \leq X_2 \leq 7.0\} \leq 0.623 \quad (32)$$

$$0.375 \leq P\{3.0 \leq X_1 \leq 7.5 \mid 7.0 \leq X_2 \leq 9.0\} \leq 0.750 \quad (33)$$

$$0.438 \leq P\{X_2 \leq 5.0 \mid X_1 \leq 5.0\} \leq 0.585 \quad (34)$$

## Ordinal distributions

$$1.05 \cdot P\{X_2 \leq 1.0\} \leq P\{X_1 \leq 1.0\} \quad (35)$$

$$1.05 \cdot P\{X_1 \geq 7.5\} \leq P\{X_2 \geq 7.5\} \quad (36)$$

$$0.70 \cdot P\{X_1 \leq 5.0\} \leq P\{X_2 \geq 6.0\} \quad (37)$$

$$P\{8.0 < X_1 \leq 9.0\} \leq P\{8.0 < X_2 \leq 9.0\} \quad (38)$$

$$P\{X_1 \geq 8.5\} \leq P\{7.5 \leq X_1 \leq 8.5\} \quad (39)$$

$$0.50 \cdot P\{X_1 \geq 8.5, X_2 \geq 8.5\} \leq P\{X_1 \leq 2.0, X_2 \leq 2.0\} \quad (40)$$

$$P\{X_1 \leq 2.0, X_2 \leq 2.0\} \leq P\{X_1 \geq 8.5, X_2 \geq 8.5\} \quad (41)$$

## APPENDIX B

DISCRETIZED CONSTRAINTS FOR UPPER BOUNDARY ASSOCIATED  
WITH ALTERNATIVE  $\alpha$

The following constraints are constructed by adopting a discretization scheme for the constraints in Appendix A as shown in (5.2). They form a set of feasible risks  $\mathcal{B}_\alpha = \{\mathbf{p}_\alpha : \kappa_\alpha(\mathbf{p}_\alpha) \leq 0\}$  with variables  $p_{\rho_1\rho_2}$ ,  $\rho_1, \rho_2 \in \{1, \dots, 10\}$ .

Bounded marginal distributions

$$0.180 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^2 p_{\rho_1\rho_2} \leq 0.287 \quad (1)$$

$$0.480 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^5 p_{\rho_1\rho_2} \leq 0.780 \quad (2)$$

$$0.670 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^8 p_{\rho_1\rho_2} \leq 0.995 \quad (3)$$

$$0.668 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=4}^{10} p_{\rho_1\rho_2} \leq 0.784 \quad (4)$$

$$0.335 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=7}^{10} p_{\rho_1\rho_2} \leq 0.428 \quad (5)$$

$$0.090 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1\rho_2} \leq 0.350 \quad (6)$$

$$0.080 \leq \sum_{\rho_2=1}^{10} p_{10\rho_2} \leq 0.137 \quad (7)$$

$$0.165 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.288 \quad (8)$$

$$0.205 \leq \sum_{\rho_2=1}^3 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.387 \quad (9)$$

$$0.410 \leq \sum_{\rho_2=1}^5 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.584 \quad (10)$$

$$0.603 \leq \sum_{\rho_2=1}^7 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.814 \quad (11)$$

$$0.654 \leq \sum_{\rho_2=4}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.745 \quad (12)$$

$$0.550 \leq \sum_{\rho_2=5}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.628 \quad (13)$$



## Bounded marginal distributions

$$0.350 \leq \sum_{\rho_2=7}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \leq 0.430 \quad (14)$$

$$0.150 \leq \sum_{\rho_2=9}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \leq 0.220 \quad (15)$$

$$0.080 \leq \sum_{\rho_1=1}^{10} p_{\rho_1 10} \leq 0.135 \quad (16)$$

$$0.010 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=1}^2 p_{\rho_1 \rho_2} \leq 0.045 \quad (17)$$

$$0.135 \leq \sum_{\rho_2=1}^4 \sum_{\rho_1=1}^4 p_{\rho_1 \rho_2} \leq 0.230 \quad (18)$$

$$0.238 \leq \sum_{\rho_2=1}^6 \sum_{\rho_1=1}^5 p_{\rho_1 \rho_2} \leq 0.380 \quad (19)$$

$$0.460 \leq \sum_{\rho_2=1}^8 \sum_{\rho_1=1}^7 p_{\rho_1 \rho_2} \leq 0.612 \quad (20)$$

$$0.825 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^9 p_{\rho_1 \rho_2} \leq 0.950 \quad (21)$$

$$0.120 \leq \sum_{\rho_2=4}^{10} \sum_{\rho_1=1}^3 p_{\rho_1 \rho_2} \leq 0.180 \quad (22)$$

$$0.150 \leq \sum_{\rho_2=1}^3 \sum_{\rho_1=1}^7 p_{\rho_1 \rho_2} \leq 0.250 \quad (23)$$

$$0.130 \leq \sum_{\rho_2=9}^{10} \sum_{\rho_1=1}^8 p_{\rho_1 \rho_2} \leq 0.185 \quad (24)$$

$$0.480 \leq \sum_{\rho_2=4}^{10} \sum_{\rho_1=1}^9 p_{\rho_1 \rho_2} \leq 0.585 \quad (25)$$

$$0.045 \leq \sum_{\rho_1=1}^6 p_{\rho_1 1} \leq 0.075 \quad (26)$$

$$0.032 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=8}^{10} p_{\rho_1 \rho_2} \leq 0.052 \quad (27)$$

$$0.035 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=4}^6 p_{\rho_1 \rho_2} \leq 0.048 \quad (28)$$

## Bounded conditional distributions

$$\sum_{\rho_1=2}^{10} p_{\rho_1 1} - 0.054 \sum_{\rho_2=1}^{10} \sum_{\rho_1=2}^{10} p_{\rho_1 \rho_2} \geq 0.0 \quad (29-a)$$

$$\sum_{\rho_1=2}^{10} p_{\rho_1 1} - 0.125 \sum_{\rho_2=1}^{10} \sum_{\rho_1=2}^{10} p_{\rho_1 \rho_2} \leq 0.0 \quad (29-b)$$

$$\sum_{\rho_2=6}^{10} p_{10 \rho_2} - 0.450 \sum_{\rho_2=1}^{10} p_{10 \rho_2} \geq 0.0 \quad (30-a)$$

$$\sum_{\rho_2=6}^{10} p_{10 \rho_2} - 0.850 \sum_{\rho_2=1}^{10} p_{10 \rho_2} \leq 0.0 \quad (30-b)$$

$$\sum_{\rho_2=9}^{10} \sum_{\rho_1=1}^4 p_{\rho_1 \rho_2} - 0.250 \sum_{\rho_2=9}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \geq 0.0 \quad (31-a)$$

$$\sum_{\rho_2=9}^{10} \sum_{\rho_1=1}^4 p_{\rho_1 \rho_2} - 0.480 \sum_{\rho_2=9}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \leq 0.0 \quad (31-b)$$

$$\sum_{\rho_2=5}^7 \sum_{\rho_1=1}^4 p_{\rho_1 \rho_2} - 0.320 \sum_{\rho_2=5}^7 \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \geq 0.0 \quad (32-a)$$

$$\sum_{\rho_2=5}^7 \sum_{\rho_1=1}^4 p_{\rho_1 \rho_2} - 0.623 \sum_{\rho_2=5}^7 \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \leq 0.0 \quad (32-b)$$

$$\sum_{\rho_2=7}^9 \sum_{\rho_1=3}^8 p_{\rho_1 \rho_2} - 0.375 \sum_{\rho_2=7}^9 \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \geq 0.0 \quad (33-a)$$

$$\sum_{\rho_2=7}^9 \sum_{\rho_1=3}^8 p_{\rho_1 \rho_2} - 0.750 \sum_{\rho_2=7}^9 \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \leq 0.0 \quad (33-b)$$

$$\sum_{\rho_2=1}^5 \sum_{\rho_1=1}^5 p_{\rho_1 \rho_2} - 0.438 \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^5 p_{\rho_1 \rho_2} \geq 0.0 \quad (34-a)$$

$$\sum_{\rho_2=1}^5 \sum_{\rho_1=1}^5 p_{\rho_1 \rho_2} - 0.675 \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^5 p_{\rho_1 \rho_2} \leq 0.0 \quad (34-b)$$

## Ordinal distributions

$$1.05 \sum_{\rho_1=1}^{10} p_{\rho_1 1} - \sum_{\rho_2=1}^{10} p_{1 \rho_2} \leq 0.0 \quad (35)$$

$$\sum_{\rho_2=1}^{10} \sum_{\rho_1=8}^{10} p_{\rho_1 \rho_2} - \sum_{\rho_2=8}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \leq 0.0 \quad (36)$$

$$0.70 \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^5 p_{\rho_1 \rho_2} - \sum_{\rho_2=6}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \leq 0.0 \quad (37)$$

$$\sum_{\rho_2=1}^{10} p_{9 \rho_2} - \sum_{\rho_1=1}^{10} p_{\rho_1 9} \leq 0.0 \quad (38)$$

$$\sum_{\rho_2=1}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1 \rho_2} - \sum_{\rho_2=1}^{10} \sum_{\rho_1=8}^9 p_{\rho_1 \rho_2} \leq 0.0 \quad (39)$$

$$0.50 \sum_{\rho_2=9}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1 \rho_2} - \sum_{\rho_2=1}^2 \sum_{\rho_1=1}^2 p_{\rho_1 \rho_2} \leq 0.0 \quad (40)$$

$$\sum_{\rho_2=1}^2 \sum_{\rho_1=1}^2 p_{\rho_1 \rho_2} - \sum_{\rho_2=9}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1 \rho_2} \leq 0.0 \quad (41)$$

## APPENDIX C

DISCRETIZED CONSTRAINTS FOR LOWER BOUNDARY ASSOCIATED  
WITH ALTERNATIVE  $\alpha$

The following constraints are constructed by adopting a discretization scheme for the constraints in Appendix A as shown in (5.8). The constraints identical to Appendix B are omitted.

Bounded marginal distributions

$$0.480 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^4 p_{\rho_1\rho_2} \leq 0.780 \quad (2)$$

$$0.670 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^7 p_{\rho_1\rho_2} \leq 0.995 \quad (3)$$

$$0.335 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=6}^{10} p_{\rho_1\rho_2} \leq 0.428 \quad (5)$$

$$0.090 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=8}^{10} p_{\rho_1\rho_2} \leq 0.350 \quad (6)$$

$$0.080 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1\rho_2} \leq 0.137 \quad (7)$$

$$0.205 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.387 \quad (9)$$

$$0.410 \leq \sum_{\rho_2=1}^4 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.584 \quad (10)$$

$$0.654 \leq \sum_{\rho_2=3}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.745 \quad (12)$$

$$0.550 \leq \sum_{\rho_2=4}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.628 \quad (13)$$

$$0.350 \leq \sum_{\rho_2=6}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.430 \quad (14)$$

$$0.150 \leq \sum_{\rho_2=8}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.220 \quad (15)$$

$$0.080 \leq \sum_{\rho_2=9}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.135 \quad (16)$$

$$0.135 \leq \sum_{\rho_2=1}^3 \sum_{\rho_1=1}^3 p_{\rho_1\rho_2} \leq 0.230 \quad (18)$$

## Bounded marginal distributions

$$0.238 \leq \sum_{\rho_2=1}^5 \sum_{\rho_1=1}^4 p_{\rho_1\rho_2} \leq 0.380 \quad (19)$$

$$0.460 \leq \sum_{\rho_2=1}^7 \sum_{\rho_1=1}^6 p_{\rho_1\rho_2} \leq 0.612 \quad (20)$$

$$0.825 \leq \sum_{\rho_2=1}^9 \sum_{\rho_1=1}^8 p_{\rho_1\rho_2} \leq 0.950 \quad (21)$$

$$0.120 \leq \sum_{\rho_2=3}^{10} \sum_{\rho_1=1}^3 p_{\rho_1\rho_2} \leq 0.180 \quad (22)$$

$$0.150 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=1}^6 p_{\rho_1\rho_2} \leq 0.250 \quad (23)$$

$$0.480 \leq \sum_{\rho_2=3}^{10} \sum_{\rho_1=1}^8 p_{\rho_1\rho_2} \leq 0.585 \quad (25)$$

$$0.045 \leq \sum_{\rho_1=1}^5 p_{\rho_1 1} \leq 0.075 \quad (26)$$

$$0.032 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=7}^{10} p_{\rho_1\rho_2} \leq 0.052 \quad (27)$$

$$0.035 \leq \sum_{\rho_1=3}^5 p_{1\rho_2} \leq 0.048 \quad (28)$$

## Bounded conditional distributions

$$\sum_{\rho_2=6}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1\rho_2} - 0.450 \sum_{\rho_2=1}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1\rho_2} \geq 0.0 \quad (30-a)$$

$$\sum_{\rho_2=6}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1\rho_2} - 0.850 \sum_{\rho_2=1}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1\rho_2} \leq 0.0 \quad (30-b)$$

$$\sum_{\rho_2=5}^7 \sum_{\rho_1=1}^4 p_{\rho_1\rho_2} - 0.320 \sum_{\rho_2=4}^7 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \geq 0.0 \quad (32-a)$$

$$\sum_{\rho_2=5}^7 \sum_{\rho_1=1}^4 p_{\rho_1\rho_2} - 0.623 \sum_{\rho_2=4}^7 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.0 \quad (32-b)$$

$$\sum_{\rho_2=7}^9 \sum_{\rho_1=3}^7 p_{\rho_1\rho_2} - 0.375 \sum_{\rho_2=7}^9 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \geq 0.0 \quad (33-a)$$

$$\sum_{\rho_2=7}^9 \sum_{\rho_1=3}^7 p_{\rho_1\rho_2} - 0.750 \sum_{\rho_2=7}^9 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.0 \quad (33-b)$$

## Ordinal distributions

$$\sum_{\rho_2=1}^{10} \sum_{\rho_1=7}^{10} p_{\rho_1\rho_2} - \sum_{\rho_2=7}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.0 \quad (36)$$

$$\sum_{\rho_2=1}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1\rho_2} - \sum_{\rho_2=8}^9 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.0 \quad (38)$$

$$\sum_{\rho_2=1}^{10} \sum_{\rho_1=8}^{10} p_{\rho_1\rho_2} - \sum_{\rho_2=1}^{10} \sum_{\rho_1=7}^8 p_{\rho_1\rho_2} \leq 0.0 \quad (39)$$

$$0.50 \sum_{\rho_2=8}^{10} \sum_{\rho_1=8}^{10} p_{\rho_1\rho_2} - \sum_{\rho_2=1}^2 \sum_{\rho_1=1}^2 p_{\rho_1\rho_2} \leq 0.0 \quad (40)$$

$$\sum_{\rho_2=1}^2 \sum_{\rho_1=1}^2 p_{\rho_1\rho_2} - \sum_{\rho_2=8}^{10} \sum_{\rho_1=8}^{10} p_{\rho_1\rho_2} \leq 0.0 \quad (41)$$

## APPENDIX D

CONSTRAINTS FOR WAGER ALTERNATIVE  $\beta$ 

Wager alternative  $\beta$  has random rewards at year 1 and year 2, say  $X_1$  and  $X_2$  on the probability space  $(\Omega_\beta, \mathcal{F}_\beta, P_\beta)$ . Our level of understanding for the probability measures on these rewards has the same forms as shown in Appendix A. All constraints for wager alternative  $\beta$  are described below. The numbers to the right of each line correspond to the constraint numbers.

## Bounded marginal distributions

$$0.335 \leq P\{X_1 \leq 3.4\} \leq 0.535 \quad (1)$$

$$0.345 \leq P\{X_1 \leq 4.5\} \leq 0.625 \quad (2)$$

$$0.625 \leq P\{X_1 \leq 6.7\} \leq 0.847 \quad (3)$$

$$0.853 \leq P\{X_1 \leq 8.5\} \leq 0.985 \quad (4)$$

$$0.765 \leq P\{X_1 > 1.5\} \leq 0.883 \quad (5)$$

$$0.354 \leq P\{X_1 > 5.5\} \leq 0.428 \quad (6)$$

$$0.163 \leq P\{X_1 > 7.8\} \leq 0.245 \quad (7)$$

$$0.185 \leq P\{X_2 \leq 1.5\} \leq 0.345 \quad (8)$$

$$0.395 \leq P\{X_2 \leq 3.4\} \leq 0.580 \quad (9)$$

$$0.480 \leq P\{X_2 \leq 5.0\} \leq 0.625 \quad (10)$$

$$0.760 \leq P\{X_2 \leq 7.5\} \leq 0.925 \quad (11)$$

$$0.625 \leq P\{X_2 > 2.8\} \leq 0.745 \quad (12)$$

$$0.340 \leq P\{X_2 > 5.5\} \leq 0.450 \quad (13)$$

$$0.035 \leq P\{X_2 > 5.5\} \leq 0.115 \quad (14)$$

$$0.200 \leq P\{X_1 \leq 3.5, X_2 \leq 3.5\} \leq 0.500 \quad (15)$$

## Bounded marginal distributions

$$0.415 \leq P\{X_1 \leq 6.0, X_2 \leq 6.0\} \leq 0.650 \quad (16)$$

$$0.100 \leq P\{X_1 \geq 5.5, X_2 \leq 5.0\} \leq 0.200 \quad (17)$$

$$0.080 \leq P\{X_1 \leq 5.0, X_2 > 5.0\} \leq 0.200 \quad (18)$$

$$0.030 \leq P\{4.0 \leq X_1 \leq 5.5, 0.5 \leq X_2 \leq 2.5\} \leq 0.150 \quad (19)$$

$$0.010 \leq P\{4.0 \leq X_1 \leq 8.0, 5.0 \leq X_2 \leq 6.0\} \leq 0.230 \quad (20)$$

$$0.010 \leq P\{4.0 \leq X_1 \leq 8.0, 6.5 \leq X_2 \leq 7.5\} \leq 0.200 \quad (21)$$

$$0.008 \leq P\{2.0 \leq X_1 \leq 5.0, 1.5 \leq X_2 \leq 2.5\} \leq 0.130 \quad (22)$$

$$0.012 \leq P\{5.5 \leq X_1 \leq 8.0, 1.5 \leq X_2 \leq 2.5\} \leq 0.150 \quad (23)$$

$$0.010 \leq P\{X_1 \leq 3.0, X_2 > 7.5\} \leq 0.150 \quad (24)$$

$$0.005 \leq P\{3.5 \leq X_1 \leq 7.0, X_2 > 7.5\} \leq 0.250 \quad (25)$$

$$0.005 \leq P\{8.0 \leq X_1 \leq 9.0, 1.0 < X_2 \leq 2.0\} \leq 0.050 \quad (26)$$

## Bounded conditional distributions

$$0.200 \leq P\{X_2 \leq 5.0 \mid X_1 \geq 8.0\} \leq 0.670 \quad (27)$$

$$0.025 \leq P\{8.0 \leq X_2 \leq 9.0 \mid X_1 \leq 4.0\} \leq 0.350 \quad (28)$$

$$0.100 \leq P\{4.0 \leq X_1 \leq 6.0 \mid 8.0 \leq X_2 \leq 9.0\} \leq 0.450 \quad (29)$$

$$0.080 \leq P\{5.0 \leq X_1 \leq 6.0 \mid 5.0 \leq X_2 \leq 6.0\} \leq 0.250 \quad (30)$$

$$0.090 \leq P\{X_1 \leq 2.0 \mid 1.0 < X_2 \leq 2.0\} \leq 0.350 \quad (31)$$

$$0.050 \leq P\{X_1 \geq 8.0 \mid 1.0 < X_2 \leq 2.0\} \leq 0.385 \quad (32)$$

$$0.130 \leq P\{X_1 \geq 5.0 \mid 7.0 < X_2 \leq 8.0\} \leq 0.670 \quad (33)$$

## Ordinal distributions

$$0.87 \cdot P\{X_1 \leq 4.0\} \geq P\{X_1 > 6.5\} \quad (34)$$

$$0.85 \cdot P\{X_2 \leq 4.0\} \geq P\{X_2 > 6.5\} \quad (35)$$

$$P\{0.0 < X_2 \leq 1.0\} \geq P\{1.0 < X_2 \leq 2.0\} \quad (36)$$

$$P\{4.0 < X_2 \leq 5.0\} \geq P\{5.0 < X_2 \leq 6.0\} \quad (37)$$

$$P\{6.0 < X_2 \leq 7.0\} \geq P\{7.0 < X_2 \leq 8.0\} \quad (38)$$

$$P\{6.0 < X_1 \leq 7.0\} \geq P\{7.0 < X_1 \leq 8.0\} \quad (39)$$

$$P\{4.0 < X_1 \leq 5.0\} \geq P\{5.0 < X_1 \leq 6.0\} \quad (40)$$

## APPENDIX E

DISCRETIZED CONSTRAINTS FOR UPPER BOUNDARY ASSOCIATED  
WITH ALTERNATIVE  $\beta$

The following constraints are constructed by adopting a discretization scheme for the constraints in Appendix A as shown in (5.2).

Bounded marginal distributions

$$0.335 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^4 p_{\rho_1\rho_2} \leq 0.535 \quad (1)$$

$$0.345 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^5 p_{\rho_1\rho_2} \leq 0.625 \quad (2)$$

$$0.625 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^7 p_{\rho_1\rho_2} \leq 0.847 \quad (3)$$

$$0.853 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^9 p_{\rho_1\rho_2} \leq 0.985 \quad (4)$$

$$0.765 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=3}^{10} p_{\rho_1\rho_2} \leq 0.883 \quad (5)$$

$$0.354 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=7}^{10} p_{\rho_1\rho_2} \leq 0.428 \quad (6)$$

$$0.163 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=9}^{10} p_{\rho_1\rho_2} \leq 0.245 \quad (7)$$

$$0.185 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.345 \quad (8)$$

$$0.395 \leq \sum_{\rho_2=1}^4 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.580 \quad (9)$$

$$0.480 \leq \sum_{\rho_2=1}^5 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.625 \quad (10)$$

$$0.760 \leq \sum_{\rho_2=1}^8 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.925 \quad (11)$$

$$0.625 \leq \sum_{\rho_2=4}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.745 \quad (12)$$

$$0.340 \leq \sum_{\rho_2=7}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.450 \quad (13)$$

$$0.035 \leq \sum_{\rho_1=1}^{10} p_{\rho_1 10} \leq 0.115 \quad (14)$$

$$0.200 \leq \sum_{\rho_2=1}^4 \sum_{\rho_1=1}^4 p_{\rho_1\rho_2} \leq 0.500 \quad (15)$$



## Bounded marginal distributions

$$0.415 \leq \sum_{\rho_1=1}^6 \sum_{\rho_2=1}^6 p_{\rho_1 \rho_2} \leq 0.650 \quad (16)$$

$$0.100 \leq \sum_{\rho_2=1}^5 \sum_{\rho_1=6}^{10} p_{\rho_1 \rho_2} \leq 0.200 \quad (17)$$

$$0.080 \leq \sum_{\rho_2=6}^{10} \sum_{\rho_1=1}^5 p_{\rho_1 \rho_2} \leq 0.200 \quad (18)$$

$$0.030 \leq \sum_{\rho_2=1}^3 \sum_{\rho_1=4}^6 p_{\rho_1 \rho_2} \leq 0.150 \quad (19)$$

$$0.010 \leq \sum_{\rho_2=5}^6 \sum_{\rho_1=4}^8 p_{\rho_1 \rho_2} \leq 0.230 \quad (20)$$

$$0.010 \leq \sum_{\rho_2=7}^8 \sum_{\rho_1=4}^8 p_{\rho_1 \rho_2} \leq 0.200 \quad (21)$$

$$0.008 \leq \sum_{\rho_2=2}^3 \sum_{\rho_1=2}^5 p_{\rho_1 \rho_2} \leq 0.130 \quad (22)$$

$$0.012 \leq \sum_{\rho_2=2}^3 \sum_{\rho_1=6}^8 p_{\rho_1 \rho_2} \leq 0.150 \quad (23)$$

$$0.010 \leq \sum_{\rho_2=8}^{10} \sum_{\rho_1=1}^3 p_{\rho_1 \rho_2} \leq 0.150 \quad (24)$$

$$0.005 \leq \sum_{\rho_2=8}^{10} \sum_{\rho_1=4}^7 p_{\rho_1 \rho_2} \leq 0.250 \quad (25)$$

$$0.005 \leq \sum_{\rho_1=8}^9 p_{\rho_1 2} \leq 0.050 \quad (26)$$

## Bounded conditional distributions

$$\sum_{\rho_2=1}^5 \sum_{\rho_1=8}^{10} p_{\rho_1 \rho_2} - 0.200 \sum_{\rho_2=1}^{10} \sum_{\rho_1=8}^{10} p_{\rho_1 \rho_2} \geq 0.0 \quad (27-a)$$

$$\sum_{\rho_2=1}^5 \sum_{\rho_1=8}^{10} p_{\rho_1 \rho_2} - 0.670 \sum_{\rho_2=1}^{10} \sum_{\rho_1=8}^{10} p_{\rho_1 \rho_2} \leq 0.0 \quad (27-b)$$

$$\sum_{\rho_2=8}^9 \sum_{\rho_1=1}^4 p_{\rho_1 \rho_2} - 0.025 \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^4 p_{\rho_1 \rho_2} \geq 0.0 \quad (28-a)$$

$$\sum_{\rho_2=8}^9 \sum_{\rho_1=1}^4 p_{\rho_1 \rho_2} - 0.350 \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^4 p_{\rho_1 \rho_2} \leq 0.0 \quad (28-b)$$

$$\sum_{\rho_2=8}^9 \sum_{\rho_1=4}^6 p_{\rho_1 \rho_2} - 0.100 \sum_{\rho_2=8}^9 \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \geq 0.0 \quad (29-a)$$

$$\sum_{\rho_2=8}^9 \sum_{\rho_1=4}^6 p_{\rho_1 \rho_2} - 0.450 \sum_{\rho_2=8}^9 \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \leq 0.0 \quad (29-b)$$

$$\sum_{\rho_2=5}^6 \sum_{\rho_1=5}^6 p_{\rho_1 \rho_2} - 0.080 \sum_{\rho_2=5}^6 \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \geq 0.0 \quad (30-a)$$

$$\sum_{\rho_2=5}^6 \sum_{\rho_1=5}^6 p_{\rho_1 \rho_2} - 0.250 \sum_{\rho_2=5}^6 \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \leq 0.0 \quad (30-b)$$

$$\sum_{\rho_1=1}^2 p_{\rho_1 2} - 0.090 \sum_{\rho_1=1}^{10} p_{\rho_1 2} \geq 0.0 \quad (31-a)$$

$$\sum_{\rho_1=1}^2 p_{\rho_1 2} - 0.350 \sum_{\rho_1=1}^{10} p_{\rho_1 2} \leq 0.0 \quad (31-b)$$

## Bounded conditional distributions

$$\sum_{\rho_1=8}^{10} p_{\rho_1 2} - 0.050 \sum_{\rho_1=1}^{10} p_{\rho_1 2} \geq 0.0 \quad (32\text{-a})$$

$$\sum_{\rho_1=8}^{10} p_{\rho_1 2} - 0.385 \sum_{\rho_1=1}^{10} p_{\rho_1 2} \leq 0.0 \quad (32\text{-b})$$

$$\sum_{\rho_1=5}^{10} p_{\rho_1 8} - 0.130 \sum_{\rho_1=1}^{10} p_{\rho_1 8} \geq 0.0 \quad (33\text{-a})$$

$$\sum_{\rho_1=5}^{10} p_{\rho_1 8} - 0.670 \sum_{\rho_1=1}^{10} p_{\rho_1 8} \leq 0.0 \quad (33\text{-b})$$

## Ordinal distributions

$$0.87 \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^4 p_{\rho_1 \rho_2} - \sum_{\rho_2=1}^{10} \sum_{\rho_1=7}^{10} p_{\rho_1 \rho_2} \geq 0.0 \quad (34)$$

$$0.85 \sum_{\rho_2=1}^4 \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} - \sum_{\rho_2=7}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1 \rho_2} \geq 0.0 \quad (35)$$

$$\sum_{\rho_1=1}^{10} p_{\rho_1 1} - \sum_{\rho_1=1}^{10} p_{\rho_1 2} \geq 0.0 \quad (36)$$

$$\sum_{\rho_1=1}^{10} p_{\rho_1 5} - \sum_{\rho_1=1}^{10} p_{\rho_1 6} \geq 0.0 \quad (37)$$

$$\sum_{\rho_1=1}^{10} p_{\rho_1 7} - \sum_{\rho_1=1}^{10} p_{\rho_1 8} \geq 0.0 \quad (38)$$

$$\sum_{\rho_2=1}^{10} p_{7 \rho_2} - \sum_{\rho_2=1}^{10} p_{8 \rho_2} \geq 0.0 \quad (39)$$

$$\sum_{\rho_2=1}^{10} p_{5 \rho_2} - \sum_{\rho_2=1}^{10} p_{6 \rho_2} \geq 0.0 \quad (40)$$

## APPENDIX F

DISCRETIZED CONSTRAINTS FOR LOWER BOUNDARY ASSOCIATED  
WITH ALTERNATIVE  $\beta$

Constraints identical to those in Appendix E are omitted.

Bounded marginal distributions

$$0.335 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^3 p_{\rho_1\rho_2} \leq 0.535 \quad (1)$$

$$0.345 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^4 p_{\rho_1\rho_2} \leq 0.625 \quad (2)$$

$$0.625 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^6 p_{\rho_1\rho_2} \leq 0.847 \quad (3)$$

$$0.853 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^8 p_{\rho_1\rho_2} \leq 0.985 \quad (4)$$

$$0.765 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=2}^{10} p_{\rho_1\rho_2} \leq 0.883 \quad (5)$$

$$0.354 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=6}^{10} p_{\rho_1\rho_2} \leq 0.428 \quad (6)$$

$$0.163 \leq \sum_{\rho_2=1}^{10} \sum_{\rho_1=8}^{10} p_{\rho_1\rho_2} \leq 0.245 \quad (7)$$

$$0.185 \leq \sum_{\rho_1=1}^{10} p_{\rho_1 1} \leq 0.345 \quad (8)$$

$$0.395 \leq \sum_{\rho_2=1}^3 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.580 \quad (9)$$

$$0.760 \leq \sum_{\rho_2=1}^7 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.925 \quad (11)$$

$$0.625 \leq \sum_{\rho_2=3}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.745 \quad (12)$$

$$0.340 \leq \sum_{\rho_2=6}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.450 \quad (13)$$

$$0.035 \leq \sum_{\rho_2=9}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \leq 0.115 \quad (14)$$

$$0.200 \leq \sum_{\rho_2=1}^3 \sum_{\rho_1=1}^3 p_{\rho_1\rho_2} \leq 0.500 \quad (15)$$

$$0.100 \leq \sum_{\rho_2=1}^5 \sum_{\rho_1=5}^{10} p_{\rho_1\rho_2} \leq 0.200 \quad (17)$$

$$0.030 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=4}^5 p_{\rho_1\rho_2} \leq 0.150 \quad (19)$$

## Bounded marginal distributions

$$0.010 \leq \sum_{\rho_2=6}^7 \sum_{\rho_1=4}^8 p_{\rho_1\rho_2} \leq 0.200 \quad (21)$$

$$0.008 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=2}^5 p_{\rho_1\rho_2} \leq 0.130 \quad (22)$$

$$0.012 \leq \sum_{\rho_2=1}^2 \sum_{\rho_1=5}^8 p_{\rho_1\rho_2} \leq 0.150 \quad (23)$$

$$0.010 \leq \sum_{\rho_2=7}^{10} \sum_{\rho_1=1}^3 p_{\rho_1\rho_2} \leq 0.150 \quad (24)$$

$$0.005 \leq \sum_{\rho_2=7}^{10} \sum_{\rho_1=3}^7 p_{\rho_1\rho_2} \leq 0.250 \quad (25)$$

## Ordinal distributions

$$0.87 \sum_{\rho_2=1}^{10} \sum_{\rho_1=1}^4 p_{\rho_1\rho_2} - \sum_{\rho_2=1}^{10} \sum_{\rho_1=6}^{10} p_{\rho_1\rho_2} \geq 0.0 \quad (34)$$

$$0.85 \sum_{\rho_2=1}^4 \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} - \sum_{\rho_2=6}^{10} \sum_{\rho_1=1}^{10} p_{\rho_1\rho_2} \geq 0.0 \quad (35)$$

## APPENDIX G

## SOLUTIONS VIA THE MCMC OPTIMIZATION ALGORITHM

The following expected utilities indicate upper and lower boundaries,  $a_\alpha$  and  $d_\alpha$ , for interval  $I_\alpha = [l_\alpha, u_\alpha]$  such that  $l_\alpha \in [a_\alpha, \cdot]$  and  $u_\alpha \in [\cdot, d_\alpha]$ . The obtained variable values represent a probability mass function that maps into boundaries  $a_\alpha$  and  $d_\alpha$  for the set of expected utilities.

Upper boundary $d_\alpha$ : 11.0445 for alternative $\alpha$ with $p_{\rho_1\rho_2}$ :					
$\rho_1 \backslash \rho_2$	1	2	3	4	5
1	0.0	0.0	0.0	0.039000	0.0
2	0.045000	0.0	0.051000	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.062000	0.0380952	0.0
6	0.0	0.035000	0.0	0.0	0.0
7	0.0	0.033000	0.0	0.0169048	0.0
8	0.0062334	0.0457666	0.0	0.0	0.0
9	0.0	0.0	0.0	0.0	0.0
10	0.0	0.0	0.0	0.0	0.003800

$\rho_1 \backslash \rho_2$	6	7	8	9	10
1	0.0122334	0.0	0.0	0.062400	0.0
2	0.0327666	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.019000	0.0
5	0.0999048	0.0	0.0	0.016080	0.0
6	0.0	0.0	0.048000	0.0	0.009000
7	0.0	0.0	0.0	0.0	0.0
8	0.0150952	0.0	0.0199848	0.049920	0.0
9	0.0	0.0	0.104095	0.0	0.0
10	0.0	0.004080	0.004920	0.0	0.090000

Lower boundary  $a_\alpha$ : 8.4529 for alternative  $\alpha$  with  $p_{\rho_1\rho_2}$ :

$\rho_1 \backslash \rho_2$	1	2	3	4	5
1	0.040000	0.0	0.075600	0.0	0.0
2	0.0	0.005000	0.0	0.0	0.0
3	0.035000	0.072000	0.0	0.0	0.0
4	0.0	0.073500	0.0582571	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0
6	0.024500	0.0	0.0	0.0552875	0.0
7	0.038000	0.0	0.0281429	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.0647125	0.0
10	0.0	0.0	0.0	0.0	0.0

$\rho_1 \backslash \rho_2$	6	7	8	9	10
1	0.0	0.0	0.042000	0.062400	0.0
2	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0
4	0.135814	0.0	0.0	0.0	0.0
5	0.0	0.0147146	0.0	0.0	0.0
6	0.0	0.0176979	0.0	0.0342289	0.0
7	0.027486	0.0	0.0	0.0333711	0.0
8	0.0	0.0	0.0	0.0	0.0
9	0.0	0.00222875	0.040000	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0

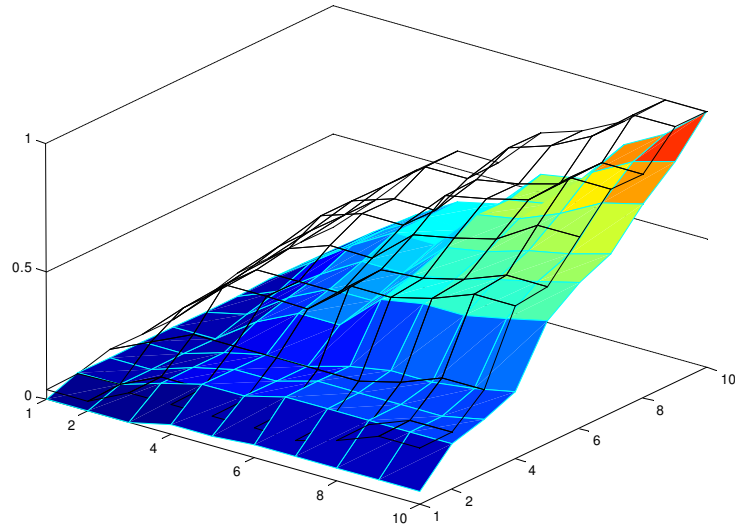


Fig. 6. Cumulative distributions of the solutions for alternative  $\alpha$ .

Upper boundary  $d_\beta$ : 10.6313 for alternative  $\beta$  with  $p_{\rho_1\rho_2}$ :

$\rho_1 \backslash \rho_2$	1	2	3	4	5
1	0.0	0.0	0.0	0.0	0.0
2	0.0	0.008325	0.0316241	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0
4	0.0205932	0.0	0.0	0.274412	0.0
5	0.0	0.0	0.0383759	0.0	0.0
6	0.0	0.040023	0.0	0.0	0.0
7	0.0	0.00853951	0.0	0.0	0.0
8	0.0719068	0.0	0.0	0.0	0.0
9	0.0	0.005000	0.0	0.0	0.0102941
10	0.0	0.0306125	0.0	0.0	0.0

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$\rho_1 \backslash \rho_2$	6	7	8	9	10
1	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0770509	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0799491	0.0
5	0.00164706	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0
7	0.0	0.010000	0.0	0.0	0.0729605
8	0.00864706	0.0	0.0	0.0109461	0.0
9	0.0	0.0	0.0279491	0.0341048	0.020652
10	0.0	0.095000	0.0	0.0	0.0213875



Lower boundary  $a_\beta$ : 7.3036 for alternative  $\beta$  with  $p_{\rho_1\rho_2}$ :

$\rho_1 \backslash \rho_2$	1	2	3	4	5
1	0.100500	0.010500	0.124000	0.0	0.0
2	0.100000	0.0	0.0	0.0	0.045000
3	0.0	0.0	0.0	0.0	0.0
4	0.009000	0.0	0.036000	0.0	0.0
5	0.006500	0.014500	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0
7	0.129000	0.0	0.0	0.0	0.0
8	0.0	0.0	0.045000	0.0	0.0
9	0.0	0.005000	0.0	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0

$\rho_1 \backslash \rho_2$	6	7	8	9	10
1	0.0	0.0	0.0	0.0	0.0
2	0.005000	0.147667	0.0	0.002333	0.0
3	0.0	0.0	0.0	0.0	0.0
4	0.0281333	0.0	0.013200	0.0	0.0
5	0.0	0.003667	0.0	0.0	0.0
6	0.007200	0.0174667	0.0	0.0	0.0
7	0.004667	0.0	0.0	0.032667	0.0
8	0.0	0.076200	0.026800	0.0	0.0
9	0.0	0.010000	0.0	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0

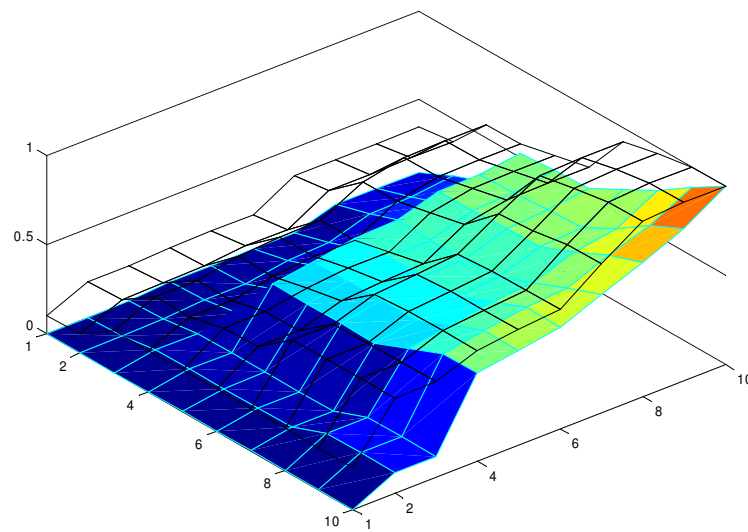


Fig. 7. Cumulative distributions of the solutions for alternative  $\beta$ .

## VITA

Jee Hyuk Park earned his Bachelor of Science and Master of Science degrees in Industrial Engineering from Kangnung National University, Kangwon-do, South Korea. Areas of concentration in his Master's program were operations research and scheduling for parallel processing systems with a thesis title of "A Study on the Genetic Algorithms for the Scheduling of Both Coarse and Fine Grain Types of Parallel Computation". He served in the Army as a Korean Augmented To US Army (KATUSA) between 1998 and 2000. After completion of his service, a project manager was his first job at SDL Alpnet Inc., Korea branch, located in Seoul, South Korea. He joined the Ph.D. degree program of Industrial Engineering, Texas A&M University in 2002 and completed his Ph.D. in 2008. His research interests include engineering wagers, applied probability and optimization algorithms.